

ON THE STICKELBERGER SPLITTING MAP IN THE K -THEORY OF NUMBER FIELDS

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ABSTRACT. The Stickelberger splitting map in the case of abelian extensions F/\mathbb{Q} was defined in [Ba1, Chap. IV]. The construction used Stickelberger's theorem. For abelian extensions F/K with an arbitrary totally real base field K the construction of [Ba1] cannot be generalized since Brumer's conjecture (the analogue of Stickelberger's theorem) is not proved yet at that level of generality. In this paper, we construct a general Stickelberger splitting map under the assumption that the first Stickelberger elements annihilate the Quillen K -groups $K_2(\mathcal{O}_{F_k})$ for the Iwasawa tower $F_{l^k} := F(\mu_{l^k})$, for $k \geq 1$. The results of [Po] give examples of CM abelian extensions F/K of general totally real base-fields K for which the first Stickelberger elements annihilate $K_2(\mathcal{O}_{F_{l^k}})_l$ for all $k \geq 1$, while this is proved in full generality in [GP], under the assumption that the Iwasawa μ -invariant $\mu_{F,l}$ vanishes. As a consequence, our Stickelberger splitting map leads to annihilation results as predicted by the original Coates-Sinnott conjecture for the subgroups $\text{div}(K_{2n}(F)_l)$ of $K_{2n}(\mathcal{O}_F)_l$ consisting of all the l -divisible elements in the even Quillen K -groups of F , for all odd primes l and all n . In §6, we construct a Stickelberger splitting map for étale K -theory. Finally, we construct both the Quillen and étale Stickelberger splitting maps under the more general assumption that for some arbitrary but fixed natural number $m > 0$, the corresponding m -th Stickelberger elements annihilate $K_{2m}(\mathcal{O}_{F_k})_l$ (respectively $K_{2m}^{\text{ét}}(\mathcal{O}_{F_k})_l$), for all k .

1. INTRODUCTION

Let F/K be an abelian CM extension of a totally real number field K . Let \mathbf{f} be the conductor of F/K and let $K_{\mathbf{f}}/K$ be the ray-class field extension with conductor \mathbf{f} . Let $G_{\mathbf{f}} := G(K_{\mathbf{f}}/K)$. For all $n \in \mathbb{Z}_{\geq 0}$, Coates [C] defined higher Stickelberger elements $\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Q}[G(F/K)]$, for integral ideals \mathbf{b} of K coprime to \mathbf{f} . Deligne and Ribet proved that $\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)]$. (See section 2 below for the detailed discussion of the Stickelberger elements and their basic properties.) In 1974, Coates and Sinnott [CS] formulated the following conjecture.

Conjecture 1.1. (*Coates-Sinnott*) $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates $K_{2n}(\mathcal{O}_F)$ for each $n \geq 1$.

This should be viewed as a higher analogue of the classical conjecture of Brumer.

Conjecture 1.2. (*Brumer*) $\Theta_0(\mathbf{b}, \mathbf{f})$ annihilates $K_0(\mathcal{O}_F)_{\text{tors}} = \text{Cl}(\mathcal{O}_F)$.

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Coates and Sinnott [CS] proved that for the base field $K = \mathbb{Q}$ the element $\Theta_1(\mathbf{b}, \mathbf{f})$ annihilates $K_2(\mathcal{O}_F)$ for F/\mathbb{Q} abelian and \mathbf{b} coprime to the order of $K_2(\mathcal{O}_F)$. Moreover, they proved that $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the l -adic étale cohomology groups $K_{2n}^{et}(\mathcal{O}_F[1/l])$ for any odd prime l , any odd n and F/\mathbb{Q} abelian CM extension. One of the ingredients used in the proof is the fact that Brumer's conjecture holds true if $K = \mathbb{Q}$. This is the classical theorem of Stickelberger. The passage from annihilation of étale cohomology to that of K -theory in the case $n = 1$ was possible due to the following theorem (see [Ta2], [Co1] and [Co2].)

Theorem 1.3. (Tate [Ta2]) *The l -adic Chern map gives a canonical isomorphism*

$$K_2(\mathcal{O}_L) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_2^{et}(\mathcal{O}_L[1/l]),$$

for any number field L and any odd prime l .

The following conjecture (generalizing Tate's theorem) is closely related to that of Coates and Sinnott.

Conjecture 1.4. (Quillen-Lichtenbaum) *For any number field L any $m \geq 1$ and any odd prime l there is a natural l -adic Chern map isomorphism*

$$(1) \quad K_m(\mathcal{O}_L) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_m^{et}(\mathcal{O}_L[1/l])$$

If the Quillen-Lichtenbaum conjecture is proved, then the l -primary part of the Coates-Sinnott conjecture is established for F/\mathbb{Q} totally real abelian, l odd and n odd, via the results of [CS] mentioned above. There is hope that recent work of Suslin, Voevodsky, Rost, Friedlander, Morel, Levine, Weibel and others will lead to a proof of the Quillen-Lichtenbaum conjecture.

A different approach towards the Coates-Sinnott conjecture was taken upon in [Ba1], in the case $K = \mathbb{Q}$. Namely, in Chap. IV loc. cit., the first author constructed the Stickelberger splitting map Λ of the boundary map ∂_F in the Quillen localization sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)_l \longrightarrow K_{2n}(F)_l \xrightarrow[\Lambda]{\partial_F} \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow 0,$$

such that $\Lambda \circ \partial_F$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. This property implies that $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\text{div}(K_{2n}(F)_l)$ of divisible elements in $K_{2n}(F)_l$, which is contained in $K_{2n}(\mathcal{O}_F)_l$ (obvious from the exact sequence above and the finiteness of $K_{2n-1}(k_v)_l$, for all v .) The construction of Λ was done without appealing to étale cohomology and the Quillen-Lichtenbaum conjecture. However, the construction in loc. cit. was based on the fact that Brumer's Conjecture is known to hold for abelian extensions of \mathbb{Q} (Stickelberger's theorem). Since Brumer's conjecture is not yet proved over arbitrary totally real base fields, the construction of Λ in loc. cit. cannot be generalized.

In this paper, we take yet another approach to the construction of the map Λ for arbitrary totally real base fields. Namely, we work under the assumption that the Stickelberger elements $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilate $K_2(\mathcal{O}_{F_k})_l$ for each k , where $F_k := F(\mu_{l^k})$ and \mathbf{f}_k is the conductor of F_k/F . At this level of generality and under this assumption, the construction of Λ uses different techniques and is more elaborate than the one in [Ba1]. In the construction, we need to operate at all levels of the Iwasawa tower F_k , for $k \geq 1$ and for every prime v . In [Ba1], the construction

at each prime v used only a certain level of the Iwasawa tower. However, our efforts pay off. Even in the particular case $K = \mathbb{Q}$, this new Stickelberger splitting map construction improves upon the results in [Ba1], where the case $l|n$ was only settled up to a factor of $l^{v_l(n)}$. This factor is completely eliminated in this paper. Moreover, it was shown in [Po] that for many examples of extensions of arbitrary totally real base fields the Stickelberger element $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ indeed annihilates $K_2(\mathcal{O}_{F_k})_l$. Also, Greither-Popescu have recently showed in [GP] that under the hypothesis that the Iwasawa μ -invariant associated to F and ℓ vanishes (a classical conjecture of Iwasawa), then $\Theta_n(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2n}^{et}(\mathcal{O}_F[1/\ell])$, for all odd primes ℓ and all odd n . In particular, if combined with Tate's theorem, this result implies that $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2(\mathcal{O}_{F_k})_l$, for all k , under the above hypothesis for F . This way, we get annihilation results of the group $\text{div}(K_{2n}(F)_l)$ for extensions F/K with arbitrary totally real base field K (see Theorems 5.4 and 5.9.)

In §6, we describe briefly the construction of the Stickelberger splitting Λ^{et} for the étale K -theory which is a direct analogue of the map Λ . The étale Stickelberger splitting map Λ^{et} has similar properties and applications as Λ . Finally, in §7, we construct both Λ and Λ^{et} under the more general assumption that for some arbitrary but fixed natural number $m > 0$, the m -th Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate $K_{2m}(\mathcal{O}_{F_k})_l$ (respectively $K_{2m}^{et}(\mathcal{O}_{F_k})_l$) for each k .

We conclude this introduction with a few paragraphs showing that the groups of divisible elements in the K -theory of number fields lie at the heart of several important conjectures in number theory, trying to justify this way our efforts to understand their Galois-module structure in terms of special values of global L -functions. In 1988, Warren Sinnott pointed out to the first author that Stickelberger's Theorem for an abelian extension F/\mathbb{Q} or, more generally, Brumer's conjecture for a CM extension F/K of a totally real number field K is equivalent to the existence of a Stickelberger splitting map Λ in the following basic exact sequence

$$0 \longrightarrow \mathcal{O}_F^\times \longrightarrow F^\times \xrightarrow[\Lambda]{\partial_F} \bigoplus_v \mathbb{Z} \longrightarrow Cl(\mathcal{O}_F) \longrightarrow 0.$$

This means that Λ is a group homomorphism, such that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_0(\mathbf{b}, \mathbf{f})$. Obviously, the above exact sequence is the lower part of the Quillen localization sequence in K -theory, since $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$, $K_1(F) = F^\times$, $K_0(k_v) = \mathbb{Z}$, $K_0(\mathcal{O}_F)_{tors} = Cl(\mathcal{O}_F)$ and Quillen's ∂_F is the direct sum of the valuation maps, just as above.

Further, by [Ba2] p. 292 we observe that for any prime $l > 2$, the annihilation of $\text{div}(K_{2n}(F)_l)$ by $\Theta_n(\mathbf{b}, \mathbf{f})$ is equivalent to the existence of a "splitting" map Λ in the following exact sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)[l^k] \longrightarrow K_{2n}(F)[l^k] \xrightarrow[\Lambda]{\partial_F} \bigoplus_v K_{2n-1}(k_v)[l^k] \longrightarrow \text{div}(K_{2n}(F)_l) \longrightarrow 0$$

such that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$, for any $k \gg 0$. Hence, the group of divisible elements $\text{div}(K_{2n}(F)_l)$ is a direct analogue of the l -primary part $Cl(\mathcal{O}_F)_l$ of the class group. Any two such "splittings" Λ differ by a homomorphism in $\text{Hom}(\bigoplus_v K_{2n-1}(k_v)[l^k], K_{2n}(\mathcal{O}_F)[l^k])$. Moreover, the Coates-Sinnott conjecture is equivalent to the existence of a "splitting" Λ , such that $\Lambda \circ \partial_F$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. If the Coates-Sinnott conjecture holds, then such a "splitting" Λ is

unique and satisfies the property that $\partial_F \circ \Lambda$ is equal to the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. This is due to the fact that $\text{div}(K_{2n}(F)_l) \subset K_{2n}(\mathcal{O}_F)_l$. Clearly, in the case $\text{div}(K_{2n}(F)_l) = K_{2n}(\mathcal{O}_F)_l$, our map Λ also has the property that $\Lambda \circ \partial_F$ equals multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. Observe that if the Quillen-Lichtenbaum conjecture holds, then by Theorem 4 in [Ba2], we have

$$\text{div}(K_{2n}(F)_l) = K_{2n}(\mathcal{O}_F)_l \Leftrightarrow \left| \frac{\prod_{v|l} w_n(F_v)}{w_n(F)} \right|_l^{-1} = 1.$$

In particular, for $F = \mathbb{Q}$ and n odd, we have $w_n(\mathbb{Q}) = w_n(\mathbb{Q}_l) = 2$. Hence, according to the Quillen-Lichtenbaum conjecture, for any $l > 2$ we should have $\text{div}(K_{2n}(\mathbb{Q})_l) = K_{2n}(\mathbb{Z})_l$.

Now, let $A := Cl(\mathbb{Z}[\mu_l])_l$ and let $A^{[i]}$ denote the eigenspace corresponding to the i -th power of the Teichmüller character $\omega : G(\mathbb{Q}(\mu_l)/\mathbb{Q}) \rightarrow (\mathbb{Z}/l\mathbb{Z})^\times$. Consider the following classical conjectures in cyclotomic field theory.

Conjecture 1.5. (*Kummer-Vandiver*)

$$A^{l-1-n} = 0 \quad \text{for all } n \text{ even and } 0 \leq n \leq l-1$$

Conjecture 1.6. (*Iwasawa*)

$$A^{l-1-n} \quad \text{is cyclic for all } n \text{ odd, such that } 1 \leq n \leq l-2$$

We can state the Kummer-Vandiver and Iwasawa conjectures in terms of divisible elements in K -theory of \mathbb{Q} (see [BG1] and [BG2]):

- (1) $A^{l-1-n} = 0 \Leftrightarrow \text{div}(K_{2n}(\mathbb{Q})_l) = 0$, for all n even, with $0 \leq n \leq l-1$.
- (2) A^{l-1-n} is cyclic $\Leftrightarrow \text{div}(K_{2n}(\mathbb{Q})_l)$ is cyclic, for all n odd, with $1 \leq n \leq l-2$.

Finally, we would like to point out that the groups of divisible elements discussed in this paper are also related to the Quillen-Lichtenbaum conjecture. Namely, by comparing the exact sequence of [Sch], Satz 8 with the exact sequence of [Ba2], Theorem 2 we conclude that the Quillen-Lichtenbaum conjecture for the K -group $K_{2n}(F)$ (for any number field F and any prime $l > 2$) holds if and only if

$$\text{div}(K_{2n}(F)_l) = K_{2n}^w(\mathcal{O}_F)_l$$

where $K_{2n}^w(\mathcal{O}_F)_l$ is the wild kernel defined in [Ba2].

2. BASIC FACTS ABOUT THE STICKELBERGER IDEALS

Let F/K be abelian CM extension of a totally real number field K . Let \mathbf{f} be the conductor of F/K and let $K_{\mathbf{f}}/K$ be the ray class field extension corresponding to \mathbf{f} . Let $G_{\mathbf{f}} := G(K_{\mathbf{f}}/K)$. Every element of $G_{\mathbf{f}}$ is the Frobenius morphism $\sigma_{\mathbf{a}}$, for some ideal \mathbf{a} of \mathcal{O}_K , coprime to the conductor \mathbf{f} . Let (\mathbf{a}, F) denote the image of $\sigma_{\mathbf{a}}$ in $G(F/K)$ via the natural surjection $G_{\mathbf{f}} \rightarrow G(F/K)$. Choose a prime number l .

With the usual notations, we let $I(\mathbf{f})/P_1(\mathbf{f})$ be the ray class group of fractional ideals in K coprime to \mathbf{f} . Let \mathbf{a} and \mathbf{a}' be two fractional ideals in $I(\mathbf{f})$. The symbol $\mathbf{a} \equiv \mathbf{a}' \pmod{\mathbf{f}}$ will mean that \mathbf{a} and \mathbf{a}' are in the same class modulo $P_1(\mathbf{f})$. For $\text{Re}(s) > 1$ consider the partial zeta function of [C], p. 291

$$(2) \quad \zeta_{\mathbf{f}}(\mathbf{a}, s) := \sum_{\mathbf{c} \equiv \mathbf{a} \pmod{\mathbf{f}}} \frac{1}{N_{\mathbf{c}}^s},$$

where the sum is taken over the integral ideals $\mathbf{c} \in \mathbf{I}(\mathbf{f})$ and $N\mathbf{c}$ denotes the usual norm of the integral ideal \mathbf{c} . The partial zeta $\zeta_{\mathbf{f}}(\mathbf{a}, s)$ can be meromorphically continued to the complex plane with a single pole at $s = 1$. For $s \in \mathbb{C} \setminus \{1\}$, consider the Stickelberger element of [C], p. 297,

$$(3) \quad \Theta_s(\mathbf{b}, \mathbf{f}) := (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \sum_{\mathbf{a}} \zeta_{\mathbf{f}}(\mathbf{a}, -s)(\mathbf{a}, F)^{-1} \in \mathbb{C}[G(F/K)]$$

where the summation is over a finite set \mathcal{S} of ideals \mathbf{a} of \mathcal{O}_K coprime to \mathbf{f} such that the Artin map

$$\mathcal{S} \longrightarrow G(K_{\mathbf{f}}/K), \quad \mathbf{a} \longrightarrow \sigma_{\mathbf{a}}$$

is bijective. The element $\Theta_s(\mathbf{b}, \mathbf{f})$ can be written in the following way

$$(4) \quad \Theta_s(\mathbf{b}, \mathbf{f}) := \sum_{\mathbf{a}} \Delta_{s+1}(\mathbf{a}, \mathbf{b}, \mathbf{f})(\mathbf{a}, F)^{-1},$$

where

$$(5) \quad \Delta_{s+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) := N\mathbf{b}^{s+1}\zeta_{\mathbf{f}}(\mathbf{a}, -s) - \zeta_{\mathbf{f}}(\mathbf{ab}, -s).$$

Arithmetically, the Stickelberger elements $\Theta_s(\mathbf{b}, \mathbf{f})$ are most interesting for values $s = n \in \mathbb{N} \cup \{0\}$. If $\mathbf{a}, \mathbf{b}, \mathbf{f}$ are integral ideals, such that \mathbf{ab} is coprime to \mathbf{f} , then Deligne and Ribet [DR] proved that $\Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f})$ are l -adic integers for all primes $l \nmid N\mathbf{b}$ and all $n \geq 0$. Moreover,

$$(6) \quad \Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv N(\mathbf{ab})^n \Delta_1(\mathbf{a}, \mathbf{b}, \mathbf{f}) \pmod{w_n(K_{\mathbf{f}})}.$$

As usual, if L is a number field, then $w_n(L)$ is the largest number $m \in \mathbb{N}$ such that the Galois group $G(L(\mu_m)/L)$ has exponent dividing n . Note that

$$w_n(L) = |H^0(G(\overline{L}/L), \mathbb{Q}/\mathbb{Z}(n))|,$$

where $\mathbb{Q}/\mathbb{Z}(n) := \oplus_l \mathbb{Q}_l/\mathbb{Z}_l(n)$. By Theorem 2.4 of [C], we have

$$\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)],$$

whenever \mathbf{b} is coprime to $w_{n+1}(F)$. The ideal of $\mathbb{Z}[G(F/K)]$ generated by the elements $\Theta_n(\mathbf{b}, \mathbf{f})$, for all integral ideals \mathbf{b} coprime to $w_{n+1}(F)$ is called the n -th Stickelberger's ideal for F/K .

When $K \subset F \subset E$ is a tower of finite abelian extensions then $\text{Res}_{E/F} : G(E/K) \rightarrow G(F/K)$ denotes the restriction map. We will also use the notation $\text{Res}_{E/F} : \mathbb{C}[G(E/K)] \rightarrow \mathbb{C}[G(F/K)]$ for the restriction map at the level of the corresponding group rings. When $\mathbf{f} \mid \mathbf{f}'$ and \mathbf{f} and \mathbf{f}' are divisible by the same prime ideals of \mathcal{O}_K then, for all \mathbf{b} coprime to \mathbf{f} , we have the following equality (see [C] Lemma 2.1, p. 292).

$$(7) \quad \text{Res}_{K_{\mathbf{f}'}/K_{\mathbf{f}}} \Theta_s(\mathbf{b}, \mathbf{f}') = \Theta_s(\mathbf{b}, \mathbf{f}).$$

If \mathbf{l} is a prime ideal of \mathcal{O}_K coprime to \mathbf{f} , then

$$(8) \quad \zeta_{\mathbf{f}}(\mathbf{a}, s) := \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \pmod{\mathbf{f}} \\ \mathbf{l} \nmid \mathbf{c}}} \frac{1}{N\mathbf{c}^s} + \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \pmod{\mathbf{f}} \\ \mathbf{l} \mid \mathbf{c}}} \frac{1}{N\mathbf{c}^s}$$

Observe that:

$$(9) \quad \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \pmod{\mathbf{f}} \\ \mathbf{l} | \mathbf{c}}} \frac{1}{N\mathbf{c}^s} = \sum_{\substack{\mathbf{a}' \pmod{\mathbf{l}\mathbf{f}} \\ \mathbf{a}' \equiv \mathbf{a} \pmod{\mathbf{f}}}} \sum_{\mathbf{c} \equiv \mathbf{a}' \pmod{\mathbf{l}\mathbf{f}}} \frac{1}{N\mathbf{c}^s} = \sum_{\substack{\mathbf{a}' \pmod{\mathbf{l}\mathbf{f}} \\ \mathbf{a}' \equiv \mathbf{a} \pmod{\mathbf{f}}}} \zeta_{\mathbf{l}\mathbf{f}}(\mathbf{a}', s)$$

Let us fix a finite \mathcal{S} of integral ideals \mathbf{a} in $I(\mathbf{f})$ as above. Observe that every class corresponding to an integral ideal \mathbf{a} modulo $P_1(\mathbf{f})$ can be written uniquely as a class $\mathbf{l}\mathbf{a}''$ modulo $P_1(\mathbf{f})$, for some \mathbf{a}'' from our set \mathcal{S} of chosen integral ideals. Since $\sigma_1 \in G(K_{\mathbf{f}}/K)$, this establishes a one-to-one correspondence between classes \mathbf{a} modulo $P_1(\mathbf{f})$ and \mathbf{a}'' modulo $P_1(\mathbf{f})$. If $\mathbf{l} | \mathbf{c}$, we put $\mathbf{c} = \mathbf{l}\mathbf{c}'$. Hence, we have the following equality.

$$(10) \quad \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \pmod{\mathbf{f}} \\ \mathbf{l} | \mathbf{c}}} \frac{1}{N\mathbf{c}^s} = \frac{1}{N\mathbf{l}^s} \sum_{\mathbf{a}'' \equiv \mathbf{c}' \pmod{\mathbf{f}}} \frac{1}{N\mathbf{c}'^s} = \frac{1}{N\mathbf{l}^s} \zeta_{\mathbf{f}}(\mathbf{a}'', s)$$

Formulas (8), (9) and (10) lead to the following equality:

$$(11) \quad \zeta_{\mathbf{f}}(\mathbf{a}, s) - \frac{1}{N\mathbf{l}^s} \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, s) = \sum_{\substack{\mathbf{a}' \pmod{\mathbf{l}\mathbf{f}} \\ \mathbf{a}' \equiv \mathbf{a} \pmod{\mathbf{f}}}} \zeta_{\mathbf{l}\mathbf{f}}(\mathbf{a}', s).$$

For all \mathbf{f} coprime to \mathbf{l} and for all \mathbf{b} coprime to $\mathbf{l}\mathbf{f}$, equality (11) gives:

$$(12) \quad \text{Res}_{K_{\mathbf{l}\mathbf{f}}/K_{\mathbf{f}}} \Theta_s(\mathbf{b}, \mathbf{l}\mathbf{f}) = (1 - (\mathbf{l}, F)^{-1} N\mathbf{l}^s) \Theta_s(\mathbf{b}, \mathbf{f})$$

Indeed we easily check that:

$$\begin{aligned} & \text{Res}_{K_{\mathbf{l}\mathbf{f}}/K_{\mathbf{f}}} (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \sum_{\mathbf{a}' \pmod{\mathbf{l}\mathbf{f}}} \zeta_{\mathbf{l}\mathbf{f}}(\mathbf{a}', -s) (\mathbf{a}', F)^{-1} = \\ & (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \sum_{\mathbf{a} \pmod{\mathbf{f}}} \sum_{\substack{\mathbf{a}' \pmod{\mathbf{l}\mathbf{f}} \\ \mathbf{a}' \equiv \mathbf{a} \pmod{\mathbf{f}}}} \zeta_{\mathbf{l}\mathbf{f}}(\mathbf{a}', -s) (\mathbf{a}, F)^{-1} = \\ & (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \sum_{\mathbf{a} \pmod{\mathbf{f}}} (\zeta_{\mathbf{f}}(\mathbf{a}, -s) - N\mathbf{l}^s \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, -s)) (\mathbf{a}, F)^{-1} = \\ & (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \left(\sum_{\mathbf{a} \pmod{\mathbf{f}}} \zeta_{\mathbf{f}}(\mathbf{a}, -s) (\mathbf{a}, F)^{-1} - (\mathbf{l}, F)^{-1} N\mathbf{l}^s \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, -s) (\mathbf{l}^{-1}\mathbf{a}, F)^{-1} \right) = \\ & (1 - (\mathbf{l}, F)^{-1} N\mathbf{l}^s) (N\mathbf{b}^{s+1} - (\mathbf{b}, F)) \sum_{\mathbf{a} \pmod{\mathbf{f}}} \zeta_{\mathbf{f}}(\mathbf{a}, -s) (\mathbf{a}, F)^{-1} \end{aligned}$$

Lemma 2.1. *Let $\mathbf{f} | \mathbf{f}'$ be ideals of \mathcal{O}_K coprime to \mathbf{b} . Then we have the following equality.*

$$(13) \quad \text{Res}_{K_{\mathbf{f}'}/K_{\mathbf{f}}} \Theta_s(\mathbf{b}, \mathbf{f}') = \left(\prod_{\substack{\mathbf{l} \nmid \mathbf{f} \\ \mathbf{l} | \mathbf{f}'}} (1 - (\mathbf{l}, F)^{-1} N\mathbf{l}^s) \right) \Theta_s(\mathbf{b}, \mathbf{f})$$

Proof. The lemma follows from (7) and (12). \square

In what follows, for any given abelian extension F/K , we consider the field extensions $F(\mu_{l^k})/K$, for all $k \geq 0$ and a fixed prime l . We let \mathbf{f}_k be the conductor of the abelian extension $F(\mu_{l^k})/K$. We suppress from the notation the explicit dependence of \mathbf{f}_k on l since the prime l is chosen once and for all in this paper.

3. BASIC FACTS ABOUT ALGEBRAIC K -THEORY

3.1. The Bockstein sequence and the Bott element. For a ring R we consider the Quillen K -groups

$$K_m(R) := \pi_m(\Omega BQP(R)) := [S^m, \Omega BQP(R)]$$

(see [Q1]) and the K -groups with coefficients

$$K_m(R, \mathbb{Z}/l^k) := \pi_m(\Omega BQP(R), \mathbb{Z}/l^k) := [M_{l^k}^m, \Omega BQP(R)]$$

defined by Browder and Karoubi [Br]. Quillen's K -groups can also be computed using Quillen's plus construction as $K_n(R) := \pi_n(BGL(R)^+)$. Any unital homomorphism of rings $\phi : R \rightarrow R'$ induces natural homomorphisms on K -groups

$$\phi_{R \setminus R'} : K_m(R, \diamond) \longrightarrow K_m(R', \diamond)$$

where $K_m(R, \diamond)$ denotes either $K_m(R)$ or $K_m(R, \mathbb{Z}/l^k)$.

Quillen K -theory and K -theory with coefficients admit product structures:

$$K_n(R, \diamond) \times K_m(R, \diamond) \xrightarrow{*} K_{m+n}(R, \diamond)$$

These induce graded ring structures on the groups $\bigoplus_{n \geq 0} K_n(R, \diamond)$.

For a topological space X , there is a Bockstein exact sequence

$$\longrightarrow \pi_{m+1}(X, \mathbb{Z}/l^k) \xrightarrow{b} \pi_m(X) \xrightarrow{l^k} \pi_m(X) \longrightarrow \pi_m(X, \mathbb{Z}/l^k) \longrightarrow$$

In particular, if we take $X := \Omega BQP(R)$, we get the Bockstein exact sequence in K -theory:

$$\longrightarrow K_{m+1}(R, \mathbb{Z}/l^k) \xrightarrow{b} K_m(R) \xrightarrow{l^k} K_m(R) \longrightarrow K_m(R, \mathbb{Z}/l^k) \longrightarrow$$

For any group G we have $BG = K(G, 1)$. Hence

$$\pi_n(BG) = \begin{cases} G & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Consequently, for a commutative group G and $X := BG$ the Bockstein map b gives an isomorphism $b : \pi_2(BG, \mathbb{Z}/l^k) \xrightarrow{\cong} G[l^k]$.

For a commutative ring with identity R we have $GL_1(R) = R^\times$. Assume that $\mu_{l^k} \subset R^\times$. Then $R^\times[l^k] = \mu_{l^k}$. Let β denote the natural composition of maps:

$$\mu_{l^k} \xrightarrow{b^{-1}} \pi_2(BGL_1(R); \mathbb{Z}/l^k) \longrightarrow \pi_2(BGL(R); \mathbb{Z}/l^k) \longrightarrow \pi_2(BGL(R)^+; \mathbb{Z}/l^k) = K_2(R, \mathbb{Z}/l^k)$$

We define the Bott element $\beta_k := \beta(\xi_{l^k}) \in K_2(R; \mathbb{Z}/l^k)$ as the image of ξ_{l^k} via β , where ξ_{l^k} is a fixed generator of μ_{l^k} . We let

$$\beta_k^{*n} := \beta_k * \cdots * \beta_k \in K_{2n}(R; \mathbb{Z}/l^k).$$

The Bott element β_k depends of course on the ring R . However, we suppress this dependence from the notation since it will be always clear where a given Bott

element lives. For example, if $\phi : R \rightarrow R'$ is a homomorphism of commutative rings containing μ_{l^k} , then it is clear from the definitions that the map

$$\phi_{R \setminus R'} : K_2(R; \mathbb{Z}/l^k) \longrightarrow K_2(R', \mathbb{Z}/l^k)$$

transports the Bott element for R into the Bott element for R' . By a slight abuse of notation, this will be written as $\phi_{R \setminus R'}(\beta_k) = \beta_k$.

Dwyer and Fiedlander [DF] constructed étale K-theory and proved that for any commutative, noetherian $\mathbb{Z}[1/l]$ -algebra R there are natural ring homomorphisms for all $l > 2$:

$$(14) \quad K_*(R) \longrightarrow K_*^{et}(R)$$

$$(15) \quad K_*(R; \mathbb{Z}/l^k) \longrightarrow K_*^{et}(R; \mathbb{Z}/l^k)$$

If R has finite \mathbb{Z}/l -cohomological dimension then there are Atiyach-Hirzebruch type spectral sequences [DF] Propositions 5.1, 5.2:

$$(16) \quad E_2^{p,-q} = H^p(R; \mathbb{Z}_l(q/2)) \Rightarrow K_{q-p}^{et}(R).$$

$$(17) \quad E_2^{p,-q} = H^p(R; \mathbb{Z}/l^k(q/2)) \Rightarrow K_{q-p}^{et}(R; \mathbb{Z}/l^k).$$

3.2. K-theory of finite fields. Let \mathbb{F}_q be the finite field with q elements. Quillen [Q3] has shown that:

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = 2m \text{ and } m > 0 \\ \mathbb{Z}/(q^m - 1)\mathbb{Z} & \text{if } n = 2m - 1 \text{ and } m > 0 \end{cases}$$

It was proven by Quillen [Q3] pp. 583-585 that for the finite field extension $i : \mathbb{F}_q \rightarrow \mathbb{F}_{q^f}$ and all $n \geq 1$ the natural map:

$$i : K_{2n-1}(\mathbb{F}_q) \rightarrow K_{2n-1}(\mathbb{F}_{q^f})$$

is injective and the transfer map:

$$N : K_{2n-1}(\mathbb{F}_{q^f}) \rightarrow K_{2n-1}(\mathbb{F}_q)$$

is surjective, where we simply write i instead of $i_{\mathbb{F}_q \setminus \mathbb{F}_{q^f}}$ and N instead of $Tr_{\mathbb{F}_{q^f}/\mathbb{F}_q}$. Moreover, Quillen [Q3] pp. 583-585 proved that

$$K_{2n-1}(\mathbb{F}_q) \cong K_{2n-1}(\mathbb{F}_{q^f})^{G(\mathbb{F}_{q^f}/\mathbb{F}_q)}$$

and that the Frobenius automorphism Fr_q (the canonical generator of $G(\mathbb{F}_{q^f}/\mathbb{F}_q)$) acts on $K_{2n-1}(\mathbb{F}_{q^f})$ via multiplication by q^n . Observe that

$$i \circ N = \sum_{i=0}^{f-1} Fr_q^i.$$

Hence

$$\text{Ker } N = K_{2n-1}(\mathbb{F}_{q^f})^{Fr_q - Id} = K_{2n-1}(\mathbb{F}_{q^f})^{q^n - 1}$$

because $\text{Ker } N$ is the kernel of multiplication by $\sum_{i=0}^{f-1} q^{ni} = \frac{q^{nf} - 1}{q^n - 1}$ in the cyclic group $K_{2n-1}(\mathbb{F}_{q^f})$. In particular, this shows that the norm map N induces the following isomorphism

$$K_{2n-1}(\mathbb{F}_{q^f})_{G(\mathbb{F}_{q^f}/\mathbb{F}_q)} \cong K_{2n-1}(\mathbb{F}_q).$$

By the Bockstein exact sequence and Quillen's results above, we observe that

$$K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{b} K_{2n-1}(\mathbb{F}_q)[l^k]$$

is an isomorphism. Hence, $K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k)$ is a cyclic group. Let us assume that $\mu_{l^k} \subset \mathbb{F}_q^\times$ (i.e. $l^k \mid q-1$.) In this case, Browder [Br] proved that the element β_k^{*n} is a generator of $K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k)$. Dwyer and Friedlander proved that there is a natural isomorphism of graded rings:

$$K_*(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_*^{et}(\mathbb{F}_q, \mathbb{Z}/l^k).$$

Assume $l^k \mid q-1$ and (by abuse of notation) let β_k also denote the image of the Bott element via the natural isomorphism:

$$K_2(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_2^{et}(\mathbb{F}_q, \mathbb{Z}/l^k).$$

Then by [DF] Theorem 5.6 multiplication with β_k induces isomorphisms:

$$\times \beta_k : K_i^{et}(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2}^{et}(\mathbb{F}_q, \mathbb{Z}/l^k),$$

$$\times \beta_k : K_i(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2}(\mathbb{F}_q, \mathbb{Z}/l^k),$$

In particular, if $l^k \mid q-1$ and $\alpha \in K_1(\mathbb{F}_q, \mathbb{Z}/l^k) = K_1(\mathbb{F}_q)/l^k$ is a generator, then the element $\alpha * \beta_k^{*n-1}$ is a generator of the cyclic group $K_{2n-1}(\mathbb{F}_q, \mathbb{Z}/l^k)$.

3.3. K -theory of number fields and rings of integers. Let F be a number field, let \mathcal{O}_F be its ring of integers and let k_v be the residue field for a prime v of \mathcal{O}_F . For a finite set of primes S of \mathcal{O}_F the ring of S -integers is denoted $\mathcal{O}_{F,S}$.

Quillen [Q2] proved that $K_n(\mathcal{O}_F)$ is a finitely generated group for every $n \geq 0$. Borel computed the ranks of the groups $K_n(\mathcal{O}_F)$ as follows:

$$K_n(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ \mathbb{Q}^{r_1+r_2-1} & \text{if } n = 1 \\ 0 & \text{if } n = 2m \text{ and } n > 0 \\ \mathbb{Q}^{r_1+r_2} & \text{if } n \equiv 1 \pmod{4} \text{ and } n \neq 1 \\ \mathbb{Q}^{r_2} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

We have the following localization exact sequences in Quillen K -theory and K -theory with coefficients [Q1].

$$\longrightarrow K_m(\mathcal{O}_F, \diamond) \longrightarrow K_m(F, \diamond) \xrightarrow{\partial_F} \bigoplus_v K_{m-1}(k_v, \diamond) \longrightarrow K_{m-1}(\mathcal{O}_F, \diamond) \longrightarrow$$

Let E/F be a finite extension. The natural maps in K -theory induced by the embedding $i : F \rightarrow E$ and $\sigma : E \rightarrow E$, for $\sigma \in G(E/F)$, will be denoted for simplicity by $i : K_m(F, \diamond) \rightarrow K_m(E, \diamond)$ and $\sigma : K_m(E, \diamond) \rightarrow K_m(E, \diamond)$. Observe that $i := i_{F \setminus E}$ and $\sigma := \sigma_{E \setminus E}$, according to the notation in section 3.1.

In addition to the natural maps $i, \sigma, \partial_F, \partial_E$, and product structures $*$ for K -theory of F and E introduced above, we have (see [Q]) the transfer map

$$Tr_{E/F} : K_m(E, \diamond) \longrightarrow K_m(F, \diamond)$$

and the reduction map

$$r_v : K_m(\mathcal{O}_{F,S}, \diamond) \longrightarrow K_m(k_v, \diamond)$$

for any prime $v \notin S$.

The maps discussed above enjoy many compatibility properties. For example, σ is naturally compatible with i , ∂_F , ∂_E , product structure $*$, $Tr_{E/F}$ and r_w and r_v . See e.g. [Ba1] for explanations of some of these compatibility properties. Let us mention below two nontrivial compatibility properties. By the result of Gillet [Gi], we have the following commutative diagrams in Quillen K -theory and K -theory with coefficients:

$$\begin{array}{ccc} K_m(F, \diamond) \times K_n(\mathcal{O}_F, \diamond) & \xrightarrow{*} & K_{m+n}(F, \diamond) \\ \downarrow \partial \times id & & \downarrow \partial \\ \bigoplus_v K_{m-1}(k_v, \diamond) \times K_n(\mathcal{O}_F, \diamond) & \xrightarrow{*} & \bigoplus_v K_{m+n-1}(k_v, \diamond) \end{array}$$

Let E/F be a finite extension unramified over a prime v of \mathcal{O}_F . Let w be a prime of \mathcal{O}_E over v . From now on, we will write $N_{w/v} := Tr_{k_w/k_v}$. The following diagram shows the compatibility of transfer with the boundary map in localization sequences for Quillen K -theory and K -theory with coefficients.

$$\begin{array}{ccc} K_m(E, \diamond) & \xrightarrow{\bigoplus_v \bigoplus_{w|v} \partial_w} & \bigoplus_v \bigoplus_{w|v} K_{m-1}(k_w, \diamond) \\ \downarrow Tr_{E/F} & & \downarrow \bigoplus_v \bigoplus_{w|v} N_{w/v} \\ K_m(F, \diamond) & \xrightarrow{\bigoplus_v \partial_v} & \bigoplus_v K_{m-1}(k_v, \diamond) \end{array}$$

Observe that $\partial_E = \bigoplus_v \bigoplus_{w|v} \partial_w$ and $\partial_F = \bigoplus_v \partial_v$.

4. CONSTRUCTION OF THE MAP Λ

Throughout the rest of the paper we assume that $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. For a prime v of \mathcal{O}_F , let k_v be its residue field and q_v the cardinality of k_v . Similarly, for any prime w of $\mathcal{O}_{F_{l^k}}$, we let k_w be its residue field. We put $E := F_{l^k}$. If $v \nmid l$, we observe that $k_w = k_v(\xi_{l^k})$, since the corresponding local field extension E_w/F_v is unramified. For any finite set S of primes in \mathcal{O}_F and any $k \geq 0$, there is an exact sequence [Q].

$$0 \longrightarrow K_2(\mathcal{O}_{F_{l^k}}) \longrightarrow K_2(\mathcal{O}_{F_{l^k, S}}) \xrightarrow{\partial} \bigoplus_{v \in S} \bigoplus_{w|v} K_1(k_w) \longrightarrow 0$$

Let $\xi_{w,k} \in K_1(k_w)_l$ be a generator of the l -torsion part of $K_1(k_w)$. Pick an element $x_{w,k} \in K_2(\mathcal{O}_{F_{l^k, S}})_l$ such that $\partial(x_{w,k}) = \xi_{w,k}$. Observe that $x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)}$ does not depend on the choice of $x_{w,k}$ since we assumed that $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2(\mathcal{O}_{F_{l^k}})$. Observe that if $\text{ord}(\xi_{w,k}) = l^a$, then $x_{w,k}^{l^a} \in K_2(\mathcal{O}_{F_{l^k}})$. Hence, $(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)})^{l^a} = (x_{w,k}^{l^a})^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} = 0$. Consequently, there is a well defined map:

$$\begin{aligned} \Lambda_1 : \bigoplus_{v \in S} \bigoplus_{w|v} K_1(k_w)_l &\longrightarrow K_2(\mathcal{O}_{F_{l^k, S}})_l, \\ \Lambda_1(\xi_{w,k}) &:= x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)}. \end{aligned}$$

Lemma 4.1. *The map Λ_1 satisfies the following property*

$$\partial \Lambda_1(\xi_{w,k}) := \xi_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)}.$$

Proof. The lemma follows immediately by compatibility of ∂ with $G(E/F)$ action. \square

Let v be a prime in \mathcal{O}_F sitting above $p \neq l$ in \mathbb{Z} . Let $S := S_v$ be the finite set of primes of \mathcal{O}_F consisting of all the primes over p . Let us fix an $n \in \mathbb{N}$. Let $k(v)$ be the natural number for which $l^{k(v)} \parallel q_v^n - 1$. Observe that if $l \mid q_v - 1$ then $k(v) = v_l(q_v - 1) + v_l(n)$ (see e.g. [Ba1, p. 336].) For $k \geq k(v)$ and $E := F(\mu_{l^k})$ let us define elements

$$\Lambda_n(\xi_{v,k}; l^k) := \text{Tr}_{E/F}(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}} \in K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k).$$

From now on, we will suppress the index n from the notation and we write $\Lambda(\xi_{v,k}; l^k)$ instead of $\Lambda_n(\xi_{v,k}; l^k)$.

Let us fix a prime sitting above v in each of the fields $F(\mu_{l^k})$, such that if $k \leq k'$ and w and w' are the fixed primes in $E := F(\mu_{l^k})$ and $E' := F(\mu_{l^{k'}})$, respectively, then w' sits above w . By the surjectivity of the transfer maps for K -theory of finite fields (see the end of section 3) we can associate to each k and the chosen prime w in $E := F(\mu_{l^k})$ a generator $\xi_{w,k}$ of $K_1(k_w)_l$, such that

$$N_{w'/w}(\xi_{w',k'}) = \xi_{w,k},$$

for all $k \leq k'$, where w and w' are the fixed primes in $E := F(\mu_{l^k})$ and $E' := F(\mu_{l^{k'}})$, respectively.

Let $r_{k'/k} : K_*(.; \mathbb{Z}/l^{k'}) \rightarrow K_*(.; \mathbb{Z}/l^k)$ be the coefficient reduction map. Recall that we put $N_{w/v} := \text{Tr}_{k_w/k_v}$ and $N_{w'/v} := \text{Tr}_{k_{w'}/k_v}$.

Lemma 4.2. *With notations as above, for every $k \leq k'$ we have*

$$r_{k'/k}(N_{w'/v}(\xi_{w',k'} * \beta_{k'}^{*n-1})) = N_{w/v}(\xi_{w,k} * \beta_k^{*n-1})$$

Proof. The formula follows by the compatibility of the elements $(\xi_{w,k})_w$ with respect to the norm maps, by the compatibility of Bott elements with respect to the coefficient reduction map $r_{k'/k}(\beta_{k'}) = \beta_k$, and by the projection formula. More precisely, we have the following equalities:

$$\begin{aligned} r_{k'/k}(N_{w'/v}(\xi_{w',k'} * \beta_{k'}^{*n-1})) &= N_{w'/v}(r_{k'/k}(\xi_{w',k'} * \beta_{k'}^{*n-1})) = N_{w'/v}(\xi_{w',k'} * \beta_{k'}^{*n-1}) = \\ &= N_{w/v}(N_{w'/w}(\xi_{w',k'} * \beta_{k'}^{*n-1})) = N_{w/v}(\xi_{w,k} * \beta_k^{*n-1}). \end{aligned}$$

\square

Lemma 4.3. *For all $k(v) \leq k \leq k'$, we have*

$$r_{k'/k}(\Lambda(\xi_{v,k'}; l^{k'})) = \Lambda(\xi_{v,k}; l^k)$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} K_2(\mathcal{O}_{E',S}) & \xrightarrow{\oplus_{w' \in S} \partial_{w'}} & \oplus_{w' \in S} K_1(k_{w'}) \\ \downarrow \text{Tr}_{E'/E} & & \downarrow \oplus_{w \in S} \oplus_{w' \mid w} N_{w'/w} \\ K_2(\mathcal{O}_{E,S}) & \xrightarrow{\oplus_{w \in S} \partial_w} & \oplus_{w \in S} K_1(k_w) \end{array}$$

It follows that we have $Tr_{E'/E}(x_{w',k'})^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} = x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)}$. Hence, if we use the projection formula again, we get

$$\begin{aligned} r_{k'/k}(Tr_{E'/F}(x_{w',k'}^{\Theta_1(\mathbf{b}, \mathbf{f}_{k'})} * \beta_{k'}^{*n-1})^{N\mathbf{b}^{n-1}}) &= Tr_{E/F}(Tr_{E'/E}(x_{w',k'}^{\Theta_1(\mathbf{b}, \mathbf{f}_{k'})} * \beta_k^{*n-1}))^{N\mathbf{b}^{n-1}} = \\ &= Tr_{E/F}(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}}. \end{aligned}$$

□

Let us introduce the following notation $N := \bigoplus_{v \in S} \bigoplus_{w|v} N_{w/v}$.

Proposition 4.4. *For every $k \geq k(v)$ we have*

$$\partial_F(\Lambda(\xi_{v,k}; l^k)) = (N(\xi_{w,k} * \beta_k^{*n-1}))^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

Proof. The proof is similar to the proofs of [Ba1 Theorem 1, pp. 336-340] and [BG1, Proposition 2, pp. 221-222]. The diagram at the end of section 3 gives the following commutative diagram of K -groups of coefficients

$$\begin{array}{ccc} K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{\partial_E} & \bigoplus_{v \in S} \bigoplus_{w|v} K_{2n-1}(k_w; \mathbb{Z}/l^k) , \\ \downarrow Tr_{E/F} & & \downarrow N \\ K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k) & \xrightarrow{\partial_F} & \bigoplus_{v \in S} K_{2n-1}(k_v; \mathbb{Z}/l^k) \end{array}$$

Hence we have $\partial_F \circ Tr_{E/F} = N \circ \partial_E$. The compatibilities of some of the natural maps mentioned in section 3 which will be used next can be expressed via the following commutative diagrams, explaining the action of the groups $G(E/K)$ and $G(F/K)$ on the K -groups with coefficients in the diagram above.

$$\begin{array}{ccc} K_{2n-2}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{r_w} & K_{2n-2}(k_w; \mathbb{Z}/l^k) \\ \downarrow \sigma_{\mathbf{a}}^{-1} & & \downarrow \sigma_{\mathbf{a}}^{-1} \\ K_{2n-2}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{r_{w\sigma_{\mathbf{a}}^{-1}}} & K_{2n-2}(k_{w\sigma_{\mathbf{a}}^{-1}}; \mathbb{Z}/l^k) \end{array}$$

The above diagram shows that

$$(18) \quad r_{w\sigma_{\mathbf{a}}^{-1}}(\beta_k^{*n-1}) = r_{w\sigma_{\mathbf{a}}^{-1}}((\beta_k^{*n-1})^{N\mathbf{a}^{n-1}\sigma_{\mathbf{a}}^{-1}}) = (r_w(\beta_k^{*n-1}))^{N\mathbf{a}^{n-1}\sigma_{\mathbf{a}}^{-1}}.$$

We can write the 1st Stickelberger element as follows

$$(19) \quad \Theta_1(\mathbf{b}, \mathbf{f}_k) = \sum'_{\mathbf{a} \bmod \mathbf{f}_k} \left(\sum_{\mathbf{c} \bmod \mathbf{f}_k, w^{\sigma_{\mathbf{c}^{-1}}}=w} \Delta_2(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \sigma_{\mathbf{c}^{-1}} \right) \sigma_{\mathbf{a}^{-1}},$$

where $\sum'_{\mathbf{a} \bmod \mathbf{f}_k}$ denotes the sum over a maximal set \mathcal{S} of ideal classes $\mathbf{a} \bmod \mathbf{f}_k$, such that the primes $w^{\sigma_{\mathbf{a}^{-1}}}$, for $\mathbf{a} \in \mathcal{S}$, are distinct. Since for every $m \geq 1$ we have

$$(20) \quad \Delta_{m+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv N\mathbf{a}^m N\mathbf{b}^m \Delta_1(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \pmod{w_m(K_{\mathbf{f}})}$$

(see [DR]), it is clear that for all $n \geq 2$ we get:

$$\begin{aligned} \Theta_n(\mathbf{b}, \mathbf{f}_k) &\equiv \\ \sum'_{\mathbf{a} \bmod \mathbf{f}_k} \left(\sum_{\mathbf{c} \bmod \mathbf{f}_k, w^{\sigma_{\mathbf{c}^{-1}}}=w} N\mathbf{a}^{n-1} N\mathbf{c}^{n-1} N\mathbf{b}^{n-1} \Delta_2(\mathbf{ac}, \mathbf{b}, \mathbf{f}_k) \sigma_{\mathbf{c}^{-1}} \right) \sigma_{\mathbf{a}^{-1}} &\pmod{w_1(K_{\mathbf{f}_k})}. \end{aligned}$$

Equalities (18) and (19), the result of Gillet [Gi] and the above congruences satisfied by Stickelberger elements lead to the following equalities.

$$\begin{aligned}
& \partial_E(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}} = \\
& = \sum_{\mathbf{a} \bmod \mathbf{f}_k} ' \xi_{w,k}^{\sum_{\mathbf{c} \bmod \mathbf{f}_k, w} \sigma_{\mathbf{c}-1} \Delta_2(\mathbf{a}\mathbf{c}, \mathbf{b}, \mathbf{f}_k) \sigma_{(\mathbf{a}\mathbf{c})^{-1}}} * (\beta_k^{*n-1})^{(N\mathbf{a}\mathbf{c})^{n-1} N\mathbf{b}^{n-1} \sigma_{(\mathbf{a}\mathbf{c})^{-1}}} = \\
& = (\xi_{w,k} * \beta_k^{*n-1})^{\sum_{\mathbf{a} \bmod \mathbf{f}_k} \sum_{\mathbf{c} \bmod \mathbf{f}_k, w} \sigma_{\mathbf{c}-1} \Delta_2(\mathbf{a}\mathbf{c}, \mathbf{b}, \mathbf{f}_k) (N\mathbf{a}\mathbf{c})^{n-1} N\mathbf{b}^{n-1} \sigma_{(\mathbf{a}\mathbf{c})^{-1}}} = \\
& = (\xi_{w,k} * \beta_k^{*n-1})^{\Theta_n(\mathbf{b}, \mathbf{f}_k)}.
\end{aligned}$$

By the first commutative diagram of this proof and the equalities above, we obtain

$$\begin{aligned}
\partial_F(\Lambda(\xi_{v,k}; l^k)) &= N(\partial_E(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}}) = N((\xi_{w,k} * \beta_k^{*n-1})^{\Theta_n(\mathbf{b}, \mathbf{f}_k)}) = \\
&= (N(\xi_{w,k} * \beta_k^{*n-1}))^{\Theta_n(\mathbf{b}, \mathbf{f})}
\end{aligned}$$

The last equality above is a result of the following well-known commutative diagram.

$$\begin{array}{ccc}
K_{2n-1}(k_w; \mathbb{Z}/l^k) & \xrightarrow{\sigma_{\mathbf{a}}^{-1}} & K_{2n-1}(k_{w\sigma_{\mathbf{a}}^{-1}}; \mathbb{Z}/l^k) \\
\downarrow N_{w/v} & & \downarrow N_{w\sigma_{\mathbf{a}}^{-1}/v\sigma_{\mathbf{a}}^{-1}} \\
K_{2n-1}(k_v; \mathbb{Z}/l^k) & \xrightarrow{\sigma_{\mathbf{a}}^{-1}} & K_{2n-1}(k_{v\sigma_{\mathbf{a}}^{-1}}; \mathbb{Z}/l^k)
\end{array}$$

□

Observe that for every $m > 0$ and every prime l , the Bockstein exact sequence and results of Quillen [Q2], [Q3] give natural isomorphisms

$$(21) \quad K_m(\mathcal{O}_{F,S})_l \cong \varprojlim_k K_m(\mathcal{O}_{F,S}; \mathbb{Z}/l^k),$$

$$(22) \quad K_m(k_v)_l \cong \varprojlim_k K_m(k_v; \mathbb{Z}/l^k).$$

We define $\Lambda(\xi_v) \in K_{2n}(\mathcal{O}_{F,S})_l$ to be the element corresponding to $(\Lambda(\xi_{v,k}; l^k))_k \in \varprojlim_k K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$ and define $\xi_v \in K_{2n-1}(k_v)_l$ to be the element corresponding to $(N(\xi_{w,k} * \beta_k^{*n-1}))_k \in \varprojlim_k K_{2n-1}(k_v; \mathbb{Z}/l^k)$ via these isomorphisms, respectively.

Proposition 4.5. *For every v such that $l \mid q_v^n - 1$ and for all $k \geq k(v)$, there are homomorphisms*

$$\Lambda_{v,l^k} : K_{2n-1}(k_v; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

satisfying the equality

$$\Lambda_{v,l^k}(N(\xi_{w,k} * \beta_k^{*n-1})) = \Lambda(\xi_{v,k}; l^k).$$

Proof. The definition of Λ_1 (see the beginning of section 4) combined with the natural isomorphism $K_1(k_w)/l^k \cong K_1(k_w; \mathbb{Z}/l^k\mathbb{Z})$ and the natural monomorphism $K_1(\mathcal{O}_{E,S})/l^k \rightarrow K_1(\mathcal{O}_{E,S}; \mathbb{Z}/l^k\mathbb{Z})$, coming from the corresponding Bockstein exact sequences, leads to the following homomorphism

$$\Lambda_1 : K_1(k_w; \mathbb{Z}/l^k\mathbb{Z}) \rightarrow K_2(\mathcal{O}_{E,S}; \mathbb{Z}/l^k\mathbb{Z}).$$

Multiplying on the target and on the source of this homomorphism with the $n - 1$ power of the Bott element and applying the natural isomorphism:

$$K_1(k_w; \mathbb{Z}/l^k \mathbb{Z}) \xrightarrow[\sim]{*\beta_k^{*n-1}} K_{2n-1}(k_w; \mathbb{Z}/l^k \mathbb{Z})$$

(see section 3) show that there exists a unique homomorphism

$$\Lambda_1 * \beta_k^{*n-1} : K_{2n-1}(k_w; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k),$$

sending $\xi_{w,k} * \beta_k^{*n-1} \rightarrow x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1}$. (Note that the Bott elements β_k showing up in the left and right of the equality above live in $K_2(k_w; \mathbb{Z}/l^k \mathbb{Z})$ and $K_2(\mathcal{O}_{E,S}; \mathbb{Z}/l^k \mathbb{Z})$, respectively.) Next, we compose the homomorphisms $\Lambda_1 * \beta_k^{*n-1}$ defined above and

$$Tr_{E/F} : K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

to obtain the following homomorphism:

$$Tr_{E/F} \circ (\Lambda_1 * \beta_k^{*n-1}) : K_{2n-1}(k_w; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k).$$

This homomorphism factors through the quotient of $G(k_w/k_v)$ -coinvariants

$$K_{2n-1}(k_w; \mathbb{Z}/l^k)_{G(k_w/k_v)} := K_{2n-1}(k_w; \mathbb{Z}/l^k) / K_{2n-1}(k_w; \mathbb{Z}/l^k)^{Fr_v - Id},$$

where $Fr_v \in G(k_w/k_v) \subseteq G(E/F)$ is the Frobenius element of the prime w over v . Since Fr_v acts via q_v^n -powers on $K_{2n-1}(k_w)$, $K_{2n-1}(k_w; \mathbb{Z}/l^k) \cong K_{2n-1}(k_w)/l^k$ (see section 3) and $k \geq k(v)$, we have

$$K_{2n-1}(k_w; \mathbb{Z}/l^k)_{G(k_w/k_v)} \cong K_{2n-1}(k_w; \mathbb{Z}/l^k)/l^{k(v)} \cong K_{2n-1}(k_w)/l^{k(v)}.$$

The obvious commutative diagram with surjective vertical morphisms (see §3)

$$\begin{array}{ccc} K_{2n-1}(k_w)/l^k & \xrightarrow{\cong} & K_{2n-1}(k_w; \mathbb{Z}/l^k) \\ \downarrow N_{w/v} & & \downarrow N_{w/v} \\ K_{2n-1}(k_v)/l^k & \xrightarrow{\cong} & K_{2n-1}(k_v; \mathbb{Z}/l^k) \end{array}$$

combined with the last isomorphism above, gives an isomorphism

$$K_{2n-1}(k_w; \mathbb{Z}/l^k)_{G(k_w/k_v)} \xrightarrow[\sim]{N_{w/v}} K_{2n-1}(k_v; \mathbb{Z}/l^k)$$

Now, the required homomorphism

$$(23) \quad \Lambda_{v,l^k} : K_{2n-1}(k_v; \mathbb{Z}/l^k) \longrightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

is defined by

$$\Lambda_{v,l^k}(x) := [Tr_{E/F} \circ (\Lambda_1 * \beta_k^{*n-1}) \circ N_{w/v}^{-1}(x)]^{N\mathbf{b}^{n-1}},$$

for all $x \in K_{2n-1}(k_v; \mathbb{Z}/l^k)$. By definition, this map sends $N(\xi_{w,k} * \beta_k^{*n-1})$ onto the element $\Lambda(\xi_{v,k}; l^k) := Tr_{E/F}(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}}$. \square

It is very easy to see that the homomorphisms Λ_{v,l^k} constructed above are compatible with the coefficient reduction maps $r_{k'/k}$, for all $k' \geq k \geq k(v)$. This permits us to construct homomorphisms

$$\Lambda_v := \varprojlim_k \Lambda_{v,l^k} : K_{2n-1}(k_v)_l \rightarrow K_{2n}(\mathcal{O}_{F,S})_l,$$

for all v as in the last proposition. Observe that $K_{2n}(\mathcal{O}_{F,S}) \subset K_{2n}(F)$, hence we can think of the maps Λ_v as

$$\Lambda_v : K_{2n-1}(k_v)_l \rightarrow K_{2n}(F)_l.$$

Definition 4.6. We define the map Λ by

$$\Lambda : \bigoplus_v K_{2n-1}(k_v)_l \rightarrow K_{2n}(F)_l$$

$$\Lambda := \Lambda_n := \prod_v \Lambda_v.$$

5. MAIN RESULTS

Theorem 5.1. *The map $\Lambda := \Lambda_n$ satisfies the following property.*

$$\partial_F \circ \Lambda(\xi_v) = \xi_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} K_{2n}(\mathcal{O}_{F,S})/l^k & \xrightarrow{\bigoplus_{v \in S} \partial_v} & \bigoplus_{v \in S} K_{2n-1}(k_v)/l^k \\ \downarrow & & \downarrow \\ K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k) & \xrightarrow{\bigoplus_{v \in S} \partial_v} & \bigoplus_{v \in S} K_{2n-1}(k_v; \mathbb{Z}/l^k) \end{array}$$

The vertical arrows in the diagram come from the Bockstein exact sequence. It is clear from the diagram that the inverse limit over k of the bottom horizontal arrow gives the boundary map $\partial_F = \bigoplus_{v \in S} \partial_v$:

$$\partial_F : K_{2n}(\mathcal{O}_{F,S})_l \rightarrow \bigoplus_v K_{2n-1}(k_v)_l.$$

Now, the theorem follows from Propositions 4.4 and 4.5. \square

In the next proposition we will construct a Stickelberger splitting map Γ which is complementary to the map Λ constructed above.

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)_l \xrightarrow{i} K_{2n}(F)_l \xleftarrow{\Gamma} K_{2n}(F)_l \xleftarrow{\Lambda} \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow 0.$$

Proposition 5.2. *The existence of the map Λ satisfying the property $\partial_F \circ \Lambda(\xi_v) = \xi_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$ is equivalent to the existence of the map $\Gamma : K_{2n}(F)_l \rightarrow K_{2n}(\mathcal{O}_F)_l$ with the property $\Gamma \circ i(\eta) = \eta^{\Theta_n(\mathbf{b}, \mathbf{f})}$. Moreover $\Gamma \circ \Lambda = 0$.*

Proof. Assume that we have the map Λ . For any $\eta \in K_{2n}(F)_l$ define

$$\Gamma(\eta) := \Lambda \circ \partial_F(\eta^{-1}) \eta^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

Observe that, in principle, we have $\Gamma(\eta) \in K_{2n}(F)_l$. However, by Theorem 5.1,

$$\partial_F(\Gamma(\eta)) = \partial_F(\Lambda \circ \partial_F(\eta^{-1}) \eta^{\Theta_n(\mathbf{b}, \mathbf{f})}) = \partial_F(\eta)^{-\Theta_n(\mathbf{b}, \mathbf{f})} \partial_F(\eta)^{\Theta_n(\mathbf{b}, \mathbf{f})} = 1.$$

Hence $\Gamma(\eta) \in K_{2n}(\mathcal{O}_F)_l$. Moreover, for $\eta \in K_{2n}(\mathcal{O}_F)_l$, we have

$$\Gamma \circ i(\eta) = \Lambda \circ \partial_F(i(\eta))^{-1} i(\eta)^{\Theta_n(\mathbf{b}, \mathbf{f})} = \eta^{\Theta_n(\mathbf{b}, \mathbf{f})}.$$

Now, assume that we have the map Γ . For any $(\xi_v) \in \bigoplus_v K_{2n-1}(k_v)_l$ define

$$\Lambda((\xi_v)) := \Gamma(\eta^{-1}) \eta^{\Theta_n(\mathbf{b}, \mathbf{f})},$$

where $\eta \in K_{2n}(F)_l$, such that $\partial_F(\eta) = (\xi_v)$. Observe that the definition does not depend on the choice of η . Indeed, for any other η' such that $\partial_F(\eta') = (\xi_v)$ we have $\eta' \eta^{-1} \in K_{2n}(\mathcal{O}_F)_l$. So $\Gamma((\eta' \eta^{-1})) (\eta'^{-1} \eta)^{\Theta_n(\mathbf{b}, \mathbf{f})} = 1$ by the property of Γ since $\eta' \eta^{-1} = i(\eta' \eta^{-1})$. It is clear that $\partial_F \circ \Lambda((\xi_v)) = \xi_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$. Moreover, by Theorem 5.1 we have the following equalities

$$\Gamma \circ \Lambda((\xi_v)) = \Lambda(\partial_F(\Lambda((\xi_v))^{-1})) \Lambda((\xi_v))^{\Theta_n(\mathbf{b}, \mathbf{f})} = (\Lambda((\xi_v)))^{-\Theta_n(\mathbf{b}, \mathbf{f})} \Lambda((\xi_v))^{\Theta_n(\mathbf{b}, \mathbf{f})} = 1$$

□

Remark 5.3. Observe that this map Λ is defined in the same way for both cases $l \nmid n$ and $l \mid n$. If restricted to the particular case $K = \mathbb{Q}$, our construction improves upon that of [Ba1]. In loc. cit., in the case $l \mid n$ the map Λ was constructed only up to a factor of $l^{v_l(n)}$.

Theorem 5.4. *Assume that the Stickelberger elements $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_2(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\text{div } K_{2n}(F)_l$ for all $n \geq 1$.*

Proof. The proof is very similar to the proof of [Ba1, Cor. 1, p. 340]. Let $d \in \text{div } K_{2n}(F)_l$. Take $m \in \mathbb{N}$ such that $d = x^{l^m}$ for some $x \in K_{2n}(F)_l$ and l^m annihilates $K_{2n}(\mathcal{O}_F)_l$. Then $\Lambda(\partial_F(x^{-1})) x^{\Theta_n(\mathbf{b}, \mathbf{f})} \in K_{2n}(\mathcal{O}_F)_l$ because

$$\partial_F(\Lambda(\partial_F(x^{-1})) x^{\Theta_n(\mathbf{b}, \mathbf{f})}) = \partial_F(x)^{-\Theta_n(\mathbf{b}, \mathbf{f})} \partial_F(x)^{\Theta_n(\mathbf{b}, \mathbf{f})} = 1$$

by Theorem 5.1 and Galois equivariance of ∂_F . Hence

$$(\Lambda(\partial_F(x^{-1})) x^{\Theta_n(\mathbf{b}, \mathbf{f})})^{l^m} = \Lambda(\partial_F(d^{-1})) d^{\Theta_n(\mathbf{b}, \mathbf{f})} = d^{\Theta_n(\mathbf{b}, \mathbf{f})} = 1$$

□

Remark 5.5. Observe that we can restrict the map Λ to the l^k -torsion part, for any $k \geq 1$. For any $k \gg 0$, there is an exact sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)[l^k] \longrightarrow K_{2n}(F)[l^k] \xrightarrow[\Lambda]{\partial_F} \bigoplus_v K_{2n-1}(k_v)[l^k] \longrightarrow \text{div}(K_{2n}(F)_l) \longrightarrow 0$$

By Theorem 5.1, we know that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. As pointed out in the Introduction, this implies the annihilation of $\text{div}(K_{2n}(F)_l)$ and consequently gives a second proof for Theorem 5.4

Let us define $F_0 := F$ and:

$$\Theta_n(\mathbf{b}, \mathbf{f}_0) = \begin{cases} \left(\prod_{l \nmid l} (1 - (\mathbf{l}, F)^{-1} N l^n) \right) \Theta_n(\mathbf{b}, \mathbf{f}) & \text{if } l \nmid \mathbf{f} \\ \Theta_n(\mathbf{b}, \mathbf{f}) & \text{if } l \mid \mathbf{f} \end{cases}$$

Hence by the formula (13) we get

$$(24) \quad \text{Res}_{F_{k+1}/F_k} \Theta_n(\mathbf{b}, \mathbf{f}_{k+1}) = \Theta_n(\mathbf{b}, \mathbf{f}_k)$$

Hence by formula (24) we can define the element

$$(25) \quad \Theta_n(\mathbf{b}, \mathbf{f}_\infty) := \varprojlim_k \Theta_n(\mathbf{b}, \mathbf{f}_k) \in \varprojlim_k \mathbb{Z}_l[G(F_k/F)].$$

Corollary 5.6. *Assume that the Stickelberger elements $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_2(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f}_k)$ annihilates the group $\operatorname{div} K_{2n}(F_k)_l$ for every $k \geq 0$ and every $n \geq 1$. In particular $\Theta_n(\mathbf{b}, \mathbf{f}_\infty)$ annihilates the group $\varinjlim_k \operatorname{div} K_{2n}(F_k)_l$ for every $n \geq 1$.*

Proof. Follows immediately from Theorem 5.4. \square

Theorem 5.7. *Let F/\mathbb{Q} be an abelian extensions of conductor f . Let an integer b be prime to $w_{n+1}(\mathbb{Q}(\mu_{lf}))|K_2(\mathcal{O}_F)_l|$. Then $\Theta_n(b, f)$ annihilates the group $\operatorname{div} K_{2n}(F)_l$ for all $n \geq 1$.*

Proof. Coates and Sinnott [CS] proved that $\Theta_1(b, f_k)$ annihilates $K_2(\mathcal{O}_{F_k})$ for all $k \geq 1$. Hence the theorem follows by Theorem 5.4. \square

Remark 5.8. Observe that Theorem 5.7 strengthens [Ba1, Cor. 1, p. 340] in the case $l \mid n$.

A much more general consequence of Theorem 5.4 above is the following.

Theorem 5.9. *Let F/K be an abelian CM extension of an arbitrary totally real number field K and let l be an odd prime. If the Iwasawa μ -invariant $\mu_{F,l}$ associated to F and l vanishes, then $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\operatorname{div}(K_{2n}(F)_l)$ for all $n \geq 1$.*

Proof. In [GP], it is shown that if $\mu_{F,l} = 0$, then $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates $K_{2n}^{et}(\mathcal{O}_F[1/l])$, for all odd n . From the definition of Iwasawa's μ -invariant one concludes right away that if $\mu_{F,l} = 0$, then $\mu_{F_k,l} = 0$, for all k . Consequently, $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2^{et}(\mathcal{O}_{F_k}[1/l])$, for all k . Now, one applies Tate's Theorem 1.3 to conclude that $\Theta_1(\mathbf{b}, \mathbf{f})$ annihilates $K_2(\mathcal{O}_{F_k})_l$, for all k . Theorem 5.4 implies the desired result. \square

Remark 5.10. Note that Theorem 5.6 above is indeed a particular case of Theorem 5.8, as $\mu_{F,l} = 0$ for all abelian extensions F/\mathbb{Q} and all primes l (according to a classical theorem of Ferrero-Washington and Sinnott.) It is a classical conjecture of Iwasawa that $\mu_{F,l} = 0$ for all number fields F and all primes l .

6. CONSTRUCTION OF THE MAP Λ^{et}

Since Quillen K -theory and étale K -theory of rings of integers and number fields enjoy many similar properties, we can construct the Stickelberger splitting map Λ^{et} in the setting of étale K -theory as well. This section consists of a brief description of the key steps of the construction of Λ^{et} . If R is either a number field L or its ring of l -integers $\mathcal{O}_{L,S}[1/l]$, Tate proved in [Ta2] that there is a natural isomorphism:

$$K_2(R)_l \xrightarrow{\cong} K_2^{et}(R).$$

Dwyer and Friedlander [DF] proved that there are natural isomorphisms

$$K_2(R; \mathbb{Z}/l^k) \xrightarrow{\cong} K_2^{et}(R; \mathbb{Z}/l^k),$$

for all $k \geq 1$. As explained in [Ba2], for any number field L and any finite set $S \subset \operatorname{Spec}(\mathcal{O}_L)$ we have the following commutative diagrams with exact rows and (surjective) Dwyer-Friedlander maps as vertical arrows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_{2n}(\mathcal{O}_L)_l & \longrightarrow & K_{2n}(\mathcal{O}_{L,S})_l & \longrightarrow & \bigoplus_{v \in S} K_{2n-1}(k_v)_l \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \cong \\
0 & \longrightarrow & K_{2n}^{et}(\mathcal{O}_L[1/l]) & \longrightarrow & K_{2n}^{et}(\mathcal{O}_{L,S}[1/l]) & \longrightarrow & \bigoplus_{v \in S} K_{2n-1}^{et}(k_v) \longrightarrow 0
\end{array}$$

For $n = 1$, the left and the middle vertical arrows in the above diagram are also isomorphisms, according to Tate's theorem.

We assumed throughout this paper that $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. Hence $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_2^{et}(\mathcal{O}_{F_{l^k}}[1/l])$ for all $k \geq 0$. Recall the construction of Λ_1 just before Lemma 4.1. In the diagram above, let $y_{w,k}$ and $\zeta_{w,k}$ denote the images of $x_{w,k}$ and $\xi_{w,k}$ via the middle vertical and right vertical arrows, respectively. Then, we define

$$\Lambda_1^{et}(\zeta_{w,k}) := y_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)}.$$

Clearly, the following diagram is commutative.

$$\begin{array}{ccc}
K_2(\mathcal{O}_{F_{l^k}, S})_l & \xleftarrow{\Lambda_1} & \bigoplus_{v \in S} \bigoplus_{w|v} K_1(k_w)_l \\
\downarrow \cong & & \downarrow \cong \\
K_2^{et}(\mathcal{O}_{F_{l^k}, S}[1/l]) & \xleftarrow{\Lambda_1^{et}} & \bigoplus_{v \in S} \bigoplus_{w|v} K_1^{et}(k_w)_l
\end{array}$$

We define elements $\Lambda^{et}(\zeta_{v,k}; l^k) \in K_{2n}^{et}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$ as follows:

$$\Lambda^{et}(\zeta_{v,k}; l^k) := \Lambda_n^{et}(\zeta_{v,k}; l^k) := Tr_{E/F}(y_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-1})^{N\mathbf{b}^{n-1}}.$$

Obviously, $\Lambda^{et}(\zeta_{v,k}; l^k)$ is the image of $\Lambda(\xi_{v,k}; l^k)$ via the Dwyer-Friedlander map. Now analogs of Lemmas 4.1, 4.2 and 4.3 and Propositions 4.4 and 4.5 hold for the étale case with Λ replaced by Λ^{et} , $\xi_{v,k}$ replaced by $\zeta_{v,k}$, and $x_{w,k}$ replaced by $y_{w,k}$ etc. Observe that the result of Gillet [Gi] for K-theory discussed in §3 is replaced by the compatibility of the Dwyer-Friedlander spectral sequence with the product structure ([DF] Proposition 5.4) and by Soulé's observation (see [So1] p. 275) that the localization sequence in étale cohomology (see [So1] p. 268) is compatible with the product by étale cohomology of $\mathcal{O}_{F,S}$. Eventually, these observations allow us to construct the map

$$\Lambda^{et} : \bigoplus_v K_{2n-1}^{et}(k_v) \rightarrow K_{2n}^{et}(F)_l$$

which is the étale analogue of our map Λ from §§4-5. Naturally, by construction, the following diagram commutes.

$$\begin{array}{ccc}
K_{2n}(F)_l & \xleftarrow{\Lambda} & \bigoplus_v K_{2n-1}(k_v)_l \\
\downarrow & & \downarrow \cong \\
K_{2n}^{et}(F)_l & \xleftarrow{\Lambda^{et}} & \bigoplus_v K_{2n-1}^{et}(k_v)_l
\end{array}$$

Moreover, the discussion above shows that we have the following étale analogue of Theorem 5.1.

Theorem 6.1. *The map Λ^{et} satisfies the following property.*

$$\partial_F^{et} \circ \Lambda^{et}(\zeta_v) = \zeta_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

The following is the étale analogue of Theorem 5.4.

Theorem 6.2. *Assume that the Stickelberger elements $\Theta_1(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_2^{et}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\text{div } K_{2n}^{et}(F)_l$ for all $n \geq 1$.*

Proof. The proof is identical to that of Theorem 5.4 with Λ replaced by Λ^{et} . One also observes that this theorem follows more directly from Theorem 5.4 since by [Ba2] Theorem 3(i) we know that $\text{div } K_{2n}(F)_l$ is isomorphic to $\text{div } K_{2n}^{et}(F)_l$ via the Dwyer-Friedlander map $K_{2n}(F)_l \rightarrow K_{2n}^{et}(F)_l$. \square

Remark 6.3. Based on Theorems 6.1 and 6.2, we can easily establish étale versions of Remark 5.5, Corollary 5.6 and Theorems 5.7 and 5.9.

7. CONSTRUCTION OF Λ AND Λ^{et} REVISITED

In this section we will generalize our approach used in §4 and construct the map $\Lambda' := \Lambda'_n$ for K_n , under the assumption that for some fixed $m > 0$ the Stickelberger element $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. Since the construction is similar to the those in §4, we will only sketch the proofs of these results.

Let L be a number field, such that $\mu_{l^k} \subset \mathcal{O}_{L,S}$. Let $i \in \mathbb{N}$ and let $m \in \mathbb{Z}$. Then, for $R = L$ or $R = \mathcal{O}_{L,S}$ there is a natural group isomorphism [DF] Theorem 5.6:

$$(26) \quad K_i^{et}(R; \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2m}^{et}(R; \mathbb{Z}/l^k)$$

which sends η to $\eta * \beta_k^{*m}$ for any $\eta \in K_i^{et}(R; \mathbb{Z}/l^k)$. If $m \geq 0$ this isomorphism is just the multiplication by β_k^{*m} . If $m < 0$ and $i + 2m > 0$, then the isomorphism (26) is the inverse to the multiplication by β_k^{*-m} isomorphism:

$$(27) \quad * \beta_k^{*-m} : K_{i+2m}^{et}(R; \mathbb{Z}/l^k) \xrightarrow{\cong} K_i^{et}(R; \mathbb{Z}/l^k).$$

Now, let us consider Quillen K -theory. If $m \geq 0$, there is a natural homomorphism

$$(28) \quad * \beta^{*m} : K_i(R; \mathbb{Z}/l^k) \rightarrow K_{i+2m}(R; \mathbb{Z}/l^k)$$

which is just a multiplication by β^{*m} . The homomorphism (28) is compatible with the isomorphism (26) via the Dwyer-Friedlander map. If $m < 0$ and $i + 2m > 0$, then take the homomorphism

$$(29) \quad t(m) : K_i(R; \mathbb{Z}/l^k) \rightarrow K_{i+2m}(R; \mathbb{Z}/l^k)$$

to be the composition of the left vertical, bottom horizontal and right vertical arrows of the following diagram.

$$\begin{array}{ccc} K_i(R; \mathbb{Z}/l^k) & \xrightarrow{t(m)} & K_{i+2m}(R; \mathbb{Z}/l^k) \\ \downarrow & & \uparrow \\ K_i^{et}(R; \mathbb{Z}/l^k) & \xrightarrow{(* \beta_k^{*-m})^{-1}} & K_{i+2m}^{et}(R; \mathbb{Z}/l^k) \end{array}$$

The left vertical arrow is the Dwyer-Friedlander map. The right vertical arrow is the Dwyer-Friedlander splitting [DF], Proposition 8.4. The Dwyer-Friedlander

splitting map is obtained as the multiplication of the inverse to the isomorphism $K_{i'}(R; \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i'}^{et}(R; \mathbb{Z}/l^k)$, for $i' = 1$ or $i' = 2$, by a nonnegative power of the Bott element $\beta_k^{*m'}$, with $m' \geq 0$ (see the proof of [DF], Proposition 8.4.)

Remark 7.1. It is clear that the Dwyer-Friedlander splitting from [DF], Proposition 8.4 is compatible with the maps $\mathbb{Z}/l^j \rightarrow \mathbb{Z}/l^{j-1}$ at the level of coefficients, for all $1 \leq j \leq k$. Consequently, the map $t(m)$ is naturally compatible with these maps. In addition, $t(m)$ is naturally compatible with the ring imbedding $R \rightarrow R'$, where $R' = L'$ or $R' = \mathcal{O}_{L',S}$ for a number field extension L'/L . Let

$$t^{et}(m) := (*\beta_k^{*-m})^{-1}.$$

It is clear from the above diagram that $t(m)$ and $t^{et}(m)$ are naturally compatible with the Dwyer-Friedlander maps.

Lemma 7.2. *Let $L = F(\mu_{l^k})$ and let $i > 0$ and $m < 0$, such that $i + 2m > 0$. Then, for $R = L$ or $R = \mathcal{O}_{L,S}$, the natural group homomorphisms $t^{et}(m)$ and $t(m)$ have the following properties:*

$$(30) \quad t^{et}(m)(\alpha)^{\sigma_{\mathbf{a}}} = t^{et}(m)(\alpha^{N\mathbf{a}^m\sigma_{\mathbf{a}}})$$

$$(31) \quad t(m)(\alpha)^{\sigma_{\mathbf{a}}} = t(m)(\alpha^{N\mathbf{a}^m\sigma_{\mathbf{a}}})$$

for $\alpha \in K_i^{et}(R; \mathbb{Z}/l^k)$ (resp. $\alpha \in K_i(R; \mathbb{Z}/l^k)$).

Lemma 7.3. *If $i \in \{1, 2\}$, $\alpha \in K_i(R; \mathbb{Z}/l^k)$ and $n + m > 0$ then*

$$(32) \quad t^{et}(m)(\alpha * \beta_k^{*n}) = \alpha * \beta_k^{*n+m}.$$

$$(33) \quad t(m)(\alpha * \beta_k^{*n}) = \alpha * \beta_k^{*n+m}.$$

Proof. The properties in Lemmas 7.2 and 7.3 follow directly from the definition of the maps $t^{et}(m)$ and $t(m)$. \square

If v is a prime of $\mathcal{O}_{L,S}$, $m < 0$ and $i + 2m > 0$, then we construct the morphism

$$(34) \quad t_v(m) : K_i(k_v; \mathbb{Z}/l^k) \rightarrow K_{i+2m}(k_v; \mathbb{Z}/l^k)$$

in the same way as we have done for $\mathcal{O}_{L,S}$ or L . Namely, $t_v(m)$ is the composition of the left vertical, bottom horizontal and right vertical arrows in the following diagram.

$$\begin{array}{ccc} K_i(k_v; \mathbb{Z}/l^k) & \xrightarrow{t_v(m)} & K_{i+2m}(k_v; \mathbb{Z}/l^k) \\ \downarrow \cong & & \uparrow \cong \\ K_i^{et}(k_v; \mathbb{Z}/l^k) & \xrightarrow{(*\beta_k^{*-m})^{-1}} & K_{i+2m}^{et}(k_v; \mathbb{Z}/l^k) \end{array}$$

The right vertical arrow is the inverse of the Dwyer-Friedlander map which, in the case of a finite field, is clearly seen to be equal to the Dwyer-Friedlander splitting map.

Similarly to $t^{et}(m)$ we can construct $t_v^{et}(m)$. We observe that the maps $t(m)$ and $t_v(m)$ are compatible with the reduction maps and the boundary maps. In other words, we have the following commutative diagrams.

$$\begin{array}{ccc}
K_i(\mathcal{O}_{L,S}; \mathbb{Z}/l^k) & \xrightarrow{r_v} & K_i(k_v; \mathbb{Z}/l^k) \\
\downarrow t(m) & & \downarrow t_v(m) \\
K_{i+2m}(\mathcal{O}_{L,S}; \mathbb{Z}/l^k) & \xrightarrow{r_v} & K_{i+2m}(k_v; \mathbb{Z}/l^k)
\end{array}$$

$$\begin{array}{ccc}
K_i(\mathcal{O}_{L,S}; \mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_{v \in S} K_{i-1}(k_v; \mathbb{Z}/l^k) \\
\downarrow t(m) & & \downarrow t_v(m) \\
K_{i+2m}(\mathcal{O}_{L,S}[1/l]; \mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_{v \in S} K_{i-1+2m}(k_v; \mathbb{Z}/l^k)
\end{array}$$

Let us point out that there are similar diagrams to the two diagrams above for étale K -theory and the maps $t^{et}(m)$ and $t_v^{et}(m)$.

As observed in the discussion above, the map $t(m)$ for $m < 0$ has the same properties as the multiplication by β^{*m} for $m \geq 0$. So, for $m < 0$ we define the symbols $\alpha * \beta^{*m} := t(m)(\alpha)$ (resp. $\alpha_v * \beta^{*m} := t_v(m)(\alpha_v)$), for $\alpha \in K_i(\mathcal{O}_L; \mathbb{Z}/l^k)$ (resp. $\alpha_v \in K_i(k_v; \mathbb{Z}/l^k)$). For $m \geq 0$, the symbol $\alpha * \beta^{*m}$ (resp. $\alpha_v * \beta^{*m}$) denotes the usual thing.

Let $m > 0$ be a natural number. Throughout the rest of this section we assume that $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. As §4, we let w denote a prime of $\mathcal{O}_{F_{l^k}}$ over a prime v of \mathcal{O}_F , such that $v \nmid l$. Put $E := F_{l^k}$. For any finite set S of primes in \mathcal{O}_F and any $k \geq 0$, there is an exact sequence [Q].

$$0 \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k}}) \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k}, S}) \xrightarrow{\partial} \bigoplus_{v \in S} \bigoplus_{w|v} K_{2m-1}(k_w) \longrightarrow 0$$

Let $\xi_{w,k} \in K_{2m-1}(k_w)_l$ be a generator of the l -torsion part of $K_{2m-1}(k_w)$. Pick an element $x_{w,k} \in K_{2m}(\mathcal{O}_{F_{l^k}, S})_l$ such that $\partial(x_{w,k}) = \xi_{w,k}$. Obviously, $x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)}$ does not depend on the choice of $x_{w,k}$ since $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$. If $\text{ord}(\xi_{w,k}) = l^a$, then $x_{w,k}^{l^a} \in K_{2m}(\mathcal{O}_{F_{l^k}})$. Hence, $(x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)})^{l^a} = (x_{w,k}^{l^a})^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} = 0$. Consequently, there is a well defined map:

$$\Lambda'_m : \bigoplus_{v \in S} \bigoplus_{w|v} K_{2m-1}(k_w)_l \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k}, S})_l,$$

$$\Lambda'_m(\xi_{w,k}) := x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)}.$$

Lemma 7.4. *The map Λ'_m satisfies the following property*

$$\partial \Lambda'_m(\xi_{w,k}) := \xi_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)}.$$

Proof. The lemma follows immediately by compatibility of ∂ with $G(E/F)$ action. \square

Let v be a prime in \mathcal{O}_F sitting above $p \neq l$ in \mathbb{Z} . Let $S := S_v$ be the finite set of primes of \mathcal{O}_F consisting of all the primes over p . Let us fix an $n \in \mathbb{N}$. Let $k(v)$ be the natural number for which $l^{k(v)} \parallel q_v^n - 1$. For $k \geq k(v)$, let us define elements:

$$\Lambda'_n(\xi_{v,k}; l^k) := \text{Tr}_{E/F}(x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-m})^{N\mathbf{b}^{n-m}} \in K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k).$$

As before, we will write $\Lambda'(\xi_{v,k}; l^k)$ instead of $\Lambda'_n(\xi_{v,k}; l^k)$.

Let us fix a prime sitting above v in each of the fields $F(\mu_{l^k})$, such that if $k \leq k'$ and w and w' are the fixed primes in $E = F(\mu_{l^k})$ and $E' := F(\mu_{l^{k'}})$, respectively, then w' sits above w . By the surjectivity of the transfer maps for K -theory of finite fields (see the end of §3), we can associate to each k and the chosen prime w in $E = F(\mu_{l^k})$ a generator $\xi_{w,k}$ of $K_{2m-1}(k_w)_l$, such that

$$N_{w'/w}(\xi_{w',k'}) = \xi_{w,k},$$

for all $k \leq k'$, where w and w' are the fixed primes in $E = F(\mu_{l^k})$ and $E' = F(\mu_{l^{k'}})$, respectively.

Lemma 7.5. *With notations as above, for every $k \leq k'$ we have*

$$r_{k'/k}(N_{w'/v}(\xi_{w',k'} * \beta_{k'}^{*n-m})) = N_{w/v}(\xi_{w,k} * \beta_k^{*n-m})$$

Proof. For $n - m \geq 0$ the proof is the same as the proof of Lemma 4.2. Assume that $n - m < 0$. Since the Dwyer-Friedlander maps commute with $N_{w/v}$ and $N_{w'/v}$, the proof is similar to that of Lemma 4.2 with only slight modifications. Namely, we use the projection formula for the negative twist in étale cohomology, since for any finite field \mathbb{F}_q with $l \nmid q$, we have natural isomorphisms coming from the Dwyer-Friedlander spectral sequence (cf. the end of §2)

$$(35) \quad K_{2j-1}^{et}(\mathbb{F}_q) \cong H^1(\mathbb{F}_q; \mathbb{Z}_l(j))$$

$$(36) \quad K_{2j-1}^{et}(\mathbb{F}_q; \mathbb{Z}/l^k) \cong H^1(\mathbb{F}_q; \mathbb{Z}/l^k(j)).$$

□

Lemma 7.6. *For all $k(v) \leq k \leq k'$, we have*

$$r_{k'/k}(\Lambda'(\xi_{v,k'}; l^{k'})) = \Lambda'(\xi_{v,k}; l^k)$$

Proof. As in the proof of Lemma 4.3, we observe that $\text{Tr}_{E'/E}(x_{w',k'})^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} = x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)}$. For $n - m \geq 0$, the proof is the same as that of Lemma 4.3. Assume that $n - m < 0$. We observe that $\text{Tr}_{E'/E}$ commutes with the Dwyer-Friedlander map. Hence $\text{Tr}_{E'/E}$ also commutes with the splitting of the Dwyer-Friedlander map since the splitting is a monomorphism. By the Dwyer-Friedlander spectral sequence for any number field L and any finite set S of prime ideals of \mathcal{O}_L containing all primes over l , we have the following isomorphism

$$(37) \quad K_{2j}^{et}(\mathcal{O}_{L,S}) \cong H^2(\mathcal{O}_{L,S}; \mathbb{Z}_l(j+1))$$

and the following exact sequence

$$(38) \quad 0 \rightarrow H^2(\mathcal{O}_{L,S}; \mathbb{Z}/l^k(j+1)) \rightarrow K_{2j}^{et}(\mathcal{O}_{L,S}; \mathbb{Z}/l^k) \rightarrow H^0(\mathcal{O}_{L,S}; \mathbb{Z}/l^k(j)) \rightarrow 0.$$

Since $x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} \in K_{2m}(\mathcal{O}_{F_k,S})$, then its image in $K_{2m}^{et}(\mathcal{O}_{F_k,S}; \mathbb{Z}/l^k)$ factors through $H^2(\mathcal{O}_{F_k,S}; \mathbb{Z}/l^k(m+1))$. Hence the proof is similar to that of Lemma 4.3 with the use of the projection formula for the negative twist in étale cohomology. □

Proposition 7.7. *For every $k \geq k(v)$, we have*

$$\partial_F(\Lambda'(\xi_{v,k}; l^k)) = (N(\xi_{w,k} * \beta_k^{*n-m}))^{\Theta_m(\mathbf{b}, \mathbf{f})}$$

Proof. The proof is similar to that of Proposition 4.4. The diagram at the end of section 3 gives the following commutative diagram of K -groups with coefficients

$$\begin{array}{ccc} K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{\partial_E} & \bigoplus_{v \in S} \bigoplus_{w|v} K_{2n-1}(k_w; \mathbb{Z}/l^k) , \\ \downarrow Tr_{E/F} & & \downarrow N \\ K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k) & \xrightarrow{\partial_F} & \bigoplus_{v \in S} K_{2n-1}(k_v; \mathbb{Z}/l^k) \end{array}$$

where $N := \bigoplus_v \bigoplus_{w|v} N_{w/v}$. Hence we have $\partial_F \circ Tr_{E/F} = N \circ \partial_E$. The compatibilities of some of the natural maps mentioned in section 3 which will be used next can be expressed via the following commutative diagrams, explaining the action of the groups $G(E/K)$ and $G(F/K)$ on the K -groups with coefficients in the diagram above. For $j > 0$ we use the following commutative diagram.

$$\begin{array}{ccc} K_{2j}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{r_w} & K_{2j}(k_w; \mathbb{Z}/l^k) \\ \downarrow \sigma_{\mathbf{a}}^{-1} & & \downarrow \sigma_{\mathbf{a}}^{-1} \\ K_{2j}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) & \xrightarrow{r_{w\sigma_{\mathbf{a}}^{-1}}} & K_{2j}(k_{w\sigma_{\mathbf{a}}^{-1}}; \mathbb{Z}/l^k) \end{array}$$

The above diagram gives the following equality:

$$(39) \quad r_{w\sigma_{\mathbf{a}}^{-1}}(\beta_k^{*n-m}) = r_{w\sigma_{\mathbf{a}}^{-1}}((\beta_k^{*n-m})^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}) = (r_w(\beta_k^{*n-m}))^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}.$$

For any $j \in \mathbb{Z}$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_{E,S}; \mathbb{Z}/l^k(j)) & \xrightarrow{r_w} & H^0(k_w; \mathbb{Z}/l^k(j)) \\ \downarrow \sigma_{\mathbf{a}}^{-1} & & \downarrow \sigma_{\mathbf{a}}^{-1} \\ H^0(\mathcal{O}_{E,S}; \mathbb{Z}/l^k(j)) & \xrightarrow{r_{w\sigma_{\mathbf{a}}^{-1}}} & H^0(k_{w\sigma_{\mathbf{a}}^{-1}}; \mathbb{Z}/l^k(j)) \end{array}$$

If $\xi_{l^k} := \exp(\frac{2\pi i}{l^k})$ is the generator of μ_{l^k} then the above diagram gives

$$(40) \quad r_{w\sigma_{\mathbf{a}}^{-1}}(\xi_{l^k}^{\otimes n-m}) = r_{w\sigma_{\mathbf{a}}^{-1}}(\xi_{l^k}^{\otimes n-m})^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}} = (r_w(\xi_{l^k}^{\otimes n-m}))^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}.$$

We can write the m -th Stickelberger element as follows

$$(41) \quad \Theta_m(\mathbf{b}, \mathbf{f}_k) = \sum'_{\mathbf{a} \bmod \mathbf{f}_k} \left(\sum_{\mathbf{c} \bmod \mathbf{f}_k, w^{\sigma_{\mathbf{c}^{-1}}} = w} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \sigma_{\mathbf{c}^{-1}} \right) \sigma_{\mathbf{a}^{-1}},$$

where $\sum'_{\mathbf{a} \bmod \mathbf{f}_k}$ denotes the sum over a maximal set \mathcal{S} of ideal classes $\mathbf{a} \bmod \mathbf{f}_k$, such that the primes $w^{\sigma_{\mathbf{a}^{-1}}}$, for $\mathbf{a} \in \mathcal{S}$, are distinct. By formula (20), for every $m \geq 1$ and $n \geq 1$ we have

$$\Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv N\mathbf{a}^{n-m} N\mathbf{b}^{n-m} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \pmod{w_{\min\{m,n\}}(K_{\mathbf{f}})}$$

(see [DR]). It is clear that for all $m \geq 1$ and $n \geq 1$ we get the following congruence mod $w_{\min\{m,n\}}(K_{\mathbf{f}_k})$.

$$\begin{aligned} \Theta_n(\mathbf{b}, \mathbf{f}_k) &\equiv \\ \sum'_{\mathbf{a} \bmod \mathbf{f}_k} \left(\sum_{\mathbf{c} \bmod \mathbf{f}_k, w^{\sigma_{\mathbf{c}^{-1}}} = w} N\mathbf{a}^{n-m} N\mathbf{c}^{n-m} N\mathbf{b}^{n-m} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}_k) \sigma_{\mathbf{c}^{-1}} \right) \sigma_{\mathbf{a}^{-1}} \end{aligned}$$

Equalities (39), (40), (41), Lemma 7.2, the result of Gillet [Gi], the compatibility of $t(n-m)$ and $t_v(n-m)$ with ∂ and the above congruences satisfied by Stickelberger elements lead in both cases $n-m \geq 0$ and $n-m < 0$ to the following equalities.

$$\begin{aligned}
& \partial_E(x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-m})^{N\mathbf{b}^{n-m}} = \\
& = \sum_{\mathbf{a} \bmod \mathbf{f}_k} ' \xi_{w,k}^{\sum_{\mathbf{c} \bmod \mathbf{f}_k, w} \sigma_{\mathbf{c}-1} = w} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}_k) \sigma_{(\mathbf{ac})-1} * (\beta_k^{*n-m})^{(N\mathbf{ac})^{n-m} N\mathbf{b}^{n-m} \sigma_{(\mathbf{ac})-1}} = \\
& = (\xi_{w,k} * \beta_k^{*n-m})^{\sum_{\mathbf{a} \bmod \mathbf{f}_k} \sum_{\mathbf{c} \bmod \mathbf{f}_k, w} \sigma_{\mathbf{c}-1} = w} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}_k) (N\mathbf{ac})^{n-m} N\mathbf{b}^{n-m} \sigma_{(\mathbf{ac})-1}} = \\
& = (\xi_{w,k} * \beta_k^{*n-m})^{\Theta_n(\mathbf{b}, \mathbf{f}_k)}.
\end{aligned}$$

We finish the proof in the same way as that of Proposition 4.4, by applying the first commutative diagram and the equalities above:

$$\begin{aligned}
\partial_F(\Lambda'(\xi_{v,k}; l^k)) &= N(\partial_E(x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-m})^{N\mathbf{b}^{n-m}}) = N((\xi_{w,k} * \beta_k^{*n-m})^{\Theta_n(\mathbf{b}, \mathbf{f}_k)}) = \\
&= (N(\xi_{w,k} * \beta_k^{*n-m}))^{\Theta_n(\mathbf{b}, \mathbf{f})}.
\end{aligned}$$

□

We define $\Lambda'(\xi_v) \in K_{2n}(\mathcal{O}_{F,S})_l$ to be the element corresponding to

$$(\Lambda'(\xi_{v,k}; l^k))_k \in \varprojlim_k K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

via the isomorphism (21). Also, we define $\xi_v \in K_{2n-1}(k_v)_l$ to be the element corresponding to

$$(N(\xi_{w,k} * \beta_k^{*n-m}))_k \in \varprojlim_k K_{2n-1}(k_v; \mathbb{Z}/l^k)$$

via the isomorphism (22).

Proposition 7.8. *For every v such that $l \mid q_v^n - 1$ and for all $k \geq k(v)$, there are homomorphisms*

$$\Lambda'_{v, l^k} : K_{2n-1}(k_v; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

satisfying the equality

$$\Lambda'_{v, l^k}(N(\xi_{w,k} * \beta_k^{*n-m})) = \Lambda'(\xi_{v,k}; l^k).$$

Proof. The definition of Λ'_m combined with the natural isomorphism $K_{2m-1}(k_w)/l^k \cong K_{2m-1}(k_w; \mathbb{Z}/l^k\mathbb{Z})$ and the natural monomorphism

$$K_{2m-1}(\mathcal{O}_{E,S})/l^k \rightarrow K_{2m-1}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k\mathbb{Z}),$$

coming from the corresponding Bockstein exact sequences, leads to the following homomorphism

$$\Lambda'_m : K_{2m-1}(k_w; \mathbb{Z}/l^k\mathbb{Z}) \rightarrow K_{2m}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k\mathbb{Z}).$$

Multiplying on the target and on the source of this homomorphism with the $n-m$ power of the Bott element if $n-m \geq 0$ (resp. applying the map $t_w(n-m)$ to the source and $t(n-m)$ to the target if $n-m < 0$) under the observation that the following map is an isomorphism:

$$K_{2m-1}(k_w; \mathbb{Z}/l^k\mathbb{Z}) \xrightarrow[\sim]{*\beta_k^{*n-m}} K_{2n-1}(k_w; \mathbb{Z}/l^k\mathbb{Z})$$

(cf. the notation of $t(j)$ and $t_w(j)$) show that there exists a unique homomorphism

$$\Lambda'_m * \beta_k^{*n-m} : K_{2n-1}(k_w; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k),$$

sending $\xi_{w,k} * \beta_k^{*n-m} \rightarrow x_{w,k}^{\Theta_m(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-m}$. Next, we compose the homomorphisms $\Lambda'_m * \beta_k^{*n-m}$ defined above and

$$Tr_{E/F} : K_{2n}(\mathcal{O}_{E,S}; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

to obtain the following homomorphism:

$$Tr_{E/F} \circ (\Lambda'_m * \beta_k^{*n-m}) : K_{2n-1}(k_w; \mathbb{Z}/l^k) \rightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k).$$

We observe that this homomorphism factors through the quotient of $G(k_w/k_v)$ -coinvariants

$$K_{2n-1}(k_w; \mathbb{Z}/l^k)_{G(k_w/k_v)} := K_{2n-1}(k_w; \mathbb{Z}/l^k) / K_{2n-1}(k_w; \mathbb{Z}/l^k)^{Fr_v - Id},$$

and arguing in the same way as in the Proposition 4.5 we get the homomorphism:

$$(42) \quad \Lambda'_{v,l^k} : K_{2n-1}(k_v; \mathbb{Z}/l^k) \longrightarrow K_{2n}(\mathcal{O}_{F,S}; \mathbb{Z}/l^k)$$

defined by

$$\Lambda'_{v,l^k}(x) := [Tr_{E/F} \circ (\Lambda_1 * \beta_k^{*n-m}) \circ N^{-1}(x)]^{N\mathbf{b}^{n-m}},$$

for all $x \in K_{2n-1}(k_v; \mathbb{Z}/l^k)$. By definition, this map sends $N(\xi_{w,k} * \beta_k^{*n-m})$ onto the element $\Lambda'(\xi_{v,k}; l^k) := Tr_{E/F}(x_{w,k}^{\Theta_1(\mathbf{b}, \mathbf{f}_k)} * \beta_k^{*n-m})^{N\mathbf{b}^{n-m}}$. \square

Since the homomorphisms Λ'_{v,l^k} are compatible with the coefficient reduction maps $r_{k'/k}$, for all $k' \geq k \geq k(v)$, we can construct homomorphisms

$$\Lambda'_v := \varprojlim_k \Lambda'_{v,l^k} : K_{2n-1}(k_v)_l \rightarrow K_{2n}(\mathcal{O}_{F,S})_l,$$

for all v , satisfying assumptions of the last proposition. Since $K_{2n}(\mathcal{O}_{F,S}) \subset K_{2n}(F)$, we get the maps:

$$\Lambda'_v : K_{2n-1}(k_v)_l \rightarrow K_{2n}(F)_l.$$

Definition 7.9. We define the map Λ' by

$$\Lambda' : \bigoplus_v K_{2n-1}(k_v)_l \rightarrow K_{2n}(F)_l$$

$$\Lambda' := \Lambda'_n := \prod_v \Lambda'_v.$$

Theorem 7.10. *The map $\Lambda' := \Lambda'_n$ satisfies the following property.*

$$\partial_F \circ \Lambda'(\xi_v) = \xi_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

Proof. The theorem follows by Propositions 7.7 and 7.8 (cf. the proof of Theorem 5.1). \square

Theorem 7.11. *Let $m > 0$ be a natural number. Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\text{div } K_{2n}(F)_l$ for every $n \geq 1$.*

Proof. The proof is very similar to the proof of Theorem 5.4. \square

Corollary 7.12. *Let $m > 0$ be a natural number. Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f}_k)$ annihilates the group $\operatorname{div} K_{2n}(F_k)_l$ for every $k \geq 0$ and every $n \geq 1$. In particular $\Theta_n(\mathbf{b}, \mathbf{f}_\infty)$ annihilates the group $\varinjlim_k \operatorname{div} K_{2n}(F_k)_l$ for every $n \geq 1$.*

Proof. Follows immediately from Theorem 7.11. \square

We can easily establish the étale K -theoretic analogues of these constructions. Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}^{et}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. In analogy with §6, we construct for every $n > 0$ the following map

$$\Lambda'^{et} : \bigoplus_v K_{2n-1}^{et}(k_v) \rightarrow K_{2n}^{et}(F)_l.$$

This is the étale analogue of our map Λ' . This construction leads immediately to the following results.

Theorem 7.13. *The map Λ'^{et} satisfies the following property.*

$$\partial_F^{et} \circ \Lambda'^{et}(\zeta_v) = \zeta_v^{\Theta_n(\mathbf{b}, \mathbf{f})}$$

Theorem 7.14. *Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}^{et}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $\operatorname{div} K_{2n}^{et}(F)_l$ for all $n \geq 1$.*

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